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SYMMETRIC GENERATION OF FINITE GROUPS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

Maria de la Luz Torres

June 2005

SYMMETRIC GENERATION OF FINITE GROUPS

A Thesis

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ABSTRACT

It is frequently the case that an infinite semi-direct product of the form 2^{*m} : N factored by a single relator gives a finite group, simple group or a sporadic simple group. The proof that a symmetric presentation gives a certain group is based on a manual double coset enumeration over N. Many of the smaller sporadic groups have been obtained for some large but not "too large" sporadic groups but only by single coset enumerations using computers. Other advantages of this manual double coset enumeration technique are to represent group elements in much more convenient shorter forms than their usual permutation representations and to find nice permutation representations for groups. In this thesis we construct, by-hand, several groups, including $U_3(3)$: 2, $L_2(13)$, $PGL_2(11)$ and $PGL_2(7)$, represent their elements in the short form (symmetric representation) and produce their permutation representations.

ACKNOWLEDGEMENTS

I would like to give a really big THANK YOU to my faculty advisor and committee chair, Dr. Zahid Hasan, for taking two years out of his busy schedule in order to help me to get to this point.

I would also like to thank my committee members, Dr. John Sarli and Dr. Gary Griffing, for taking the time to revise this thesis.

DEDICATION

To my family, especially my mother and father, Maria Torres and Jose Torres, who always believed in me. Also, to my wonderful husband, Jorge Rodriguez, who always gave me the support I needed to finish my studies and thesis.

TABLE OF CONTENTS

ABSTRACT iii

ACKNOWLEDGEMENTS iv

LIST OF TABLES ix

LIST OF FIGURES x

CHAPTER ONE: INTRODUCTION

Rationale 1

Symmetric Generation of a Group 2

Definitions 11

CHAPTER TWO: SYMMETRIC REPRESENTATION OF $L_2(11)$ ELEMENTS

Introduction 13

Double Coset Enumeration 14

Representation of Elements of $L_2(11)$ 25

Permutation Representation of an Element Whose Symmetric Representation Is Given 27

Proof of Isomorphism 30

CHAPTER THREE: SYMMETRIC REPRESENTATION OF $PGL_2(11)$ ELEMENTS

Introduction 38

Double Coset Enumeration 39

Representation of Elements of $PGL_2(11)$ 44

Permutation Representation of an Element Whose Symmetric Representation Is Given 45

Proof of Isomorphism 47

CHAPTER FOUR: SYMMETRIC REPRESENTATION OF $PGL_2(7)$ ELEMENTS

Introduction 51

Double Coset Enumeration 52

Representation of Elements of $PGL_2(7)$ 59

Permutation Representation of an Element Whose Symmetric Representation Is Given 61

Proof of Isomorphism 63

CHAPTER FIVE: SYMMETRIC REPRESENTATION OF $U_3(3) : 2$ ELEMENTS

Introduction 69

Double Coset Enumeration 70

Representation of Elements of $U_3(3) : 2$ 83

Permutation Representation of an Element Whose Symmetric Representation Is Given 84

Proof of Isomorphism 87

CHAPTER SIX: SYMMETRIC REPRESENTATION OF $L_2(13)$ ELEMENTS

Introduction 92

Double Coset Enumeration 93

Representation of Elements of $L_2(13)$ 106

Permutation Representation of an Element Whose Symmetric Representation Is Given 108

Proof of Isomorphism 112

CHAPTER SEVEN: SYMMETRIC REPRESENTATION OF $L_2(41)$ ELEMENTS

Introduction 118

Double Coset Enumeration 119

Representation of Elements of $L_2(41)$ 143

Proof of Isomorphism 146

CHAPTER EIGHT: NON-SYMMETRICALLY GENERATED GROUPS

Introduction 161

Group 2^4 : $(A_5 \times 2)$ 161

Double Coset Enumeration 162

Group $2 \times S_5$ 168

Double Coset Enumeration 169

Group 2^4 : $(S_5 \times 2)$ 173

Double Coset Enumeration 173

Group 2^4 : S_5 180

Double Coset Enumeration 181

CHAPTER NINE: SYMMETRIC REPRESENTATION OF $P\Sigma L_2(49)$ ELEMENTS

Introduction 184

Double Coset Enumeration 185

Proof of Isomorphism 209

APPENDIX A: $U_3(3) : 2$ 220

APPENDIX B: $L_2(41)$ 222

APPENDIX C: NON-SYMMETRICALLY GENERATED GROUPS 230

APPENDIX D: $P\Sigma L_2(49)$ 232

REFERENCES 245

LIST OF TABLES

Table 2.1. The Double Cosets [w] = NwN in G, Where N = A_4 20

Table 3.1. The Double Cosets [w] = NwN in G, Where N = $L_2(5)$. 41

Table 4.1. The Double Cosets [w] = NwN in G, Where N = A_4 56

Table 5.1. The Double Cosets [w] = NwN in G, Where N = $L_3(2)$ 79

Table 6.1. The Double Cosets [w] = NwN in G, Where N = A_4 100

Table 7.1. The Double Cosets [w] = NwN in G, Where N = $L_2(5)$ 129

Table 9.1. The Double Cosets [w] = NwN in G, Where N = S_5 194

LIST OF FIGURES

Figure 1.1. Cayley Graph of S_6 over A_5 9

Figure 1.2. Cayley Graph of S_{n+1} over A_n 10

Figure 2.1. Cayley Graph of $L_2(11)$ over A_4 22

Figure 3.1. Cayley Graph of $PGL_2(11)$ over $L_2(5)$ . . . 42

Figure 4.1. Cayley Graph of $PGL_2(7)$ over A_4 58

Figure 5.1. Cayley Graph of $U_3(3) : 2$ over $L_3(2)$ . . . 80

Figure 6.1. Cayley Graph of $L_2(13)$ over A_4 103

Figure 7.1. Cayley Graph of $L_2(41)$ over $L_2(5)$ 134

Figure 8.1. Cayley Graph of $2^4 : (A_5 \times 2)$ over A_5 . . . 167

Figure 8.2. Cayley Graph of $(2 \times S_5)$ over S_4 172

Figure 8.3. Cayley Graph of $2^4 : (S_5 \times 2)$ over S_5 . . . 179

Figure 8.4. Cayley Graph of $2^4 : S_5$ over S_5 182

Figure 9.1. Cayley Graph of $P\Sigma L_2(49)$ over S_5 198

CHAPTER ONE

INTRODUCTION

Rationale

We are interested in symmetric presentations for finite simple groups only. It is essential to know that a group is indeed finite, else when attempting to construct a finite Cayley Graph we may need to give up on a particular group presentation without knowing whether or not it determines a finite group. It is necessary to factor the progenitor which is written as an infinite semi-direct product, by a non-trivial relation, else the order of the progenitor will be infinite.

Double coset enumeration of G over N allows us to represent G as a group of permutations on the cosets of N. For example, the Janko group J_1 can be generated by 11 involutions that are permuted, by conjugation within J_1, by a subgroup isomorphic to $L_2(11)$. Hence, each element of J_1 can be written as a permutation of $L_2(11)$ followed by a word of at most four generators.

It is the main purpose of this thesis to find homomorphic images of the infinite semi-direct product $2^{*m} : N$, where 2^{*m} denotes m-copies of the cyclic group of

order 2, and N acts transitively on the set of generating involutions of the factors in the free product.

We will give algorithms for finding a symmetric representation for any element of G, and also for finding a permutation representation for any element of G whose symmetric representation is given. In this thesis we represent elements of various finite groups including the groups $U_3(3) : 2$, $L_2(13)$, $PGL_2(11)$, and $PGL_2(7)$.

Symmetric Generation of a Group

Let G be a group and let $T = \{t_1, t_2, \ldots, t_n\} \subseteq G$.

Let $\bar{T} = \{T_1, T_2, \ldots, T_n\}$, where $T_i = \langle t_i \rangle$, the cyclic subgroup generated by t_i.

Furthermore, define $N = N_G(\bar{T})$, the set normalizer in G of $\bar{T}$.

T is said to be a symmetric generating set for G iff

(i) $G = \langle T \rangle$

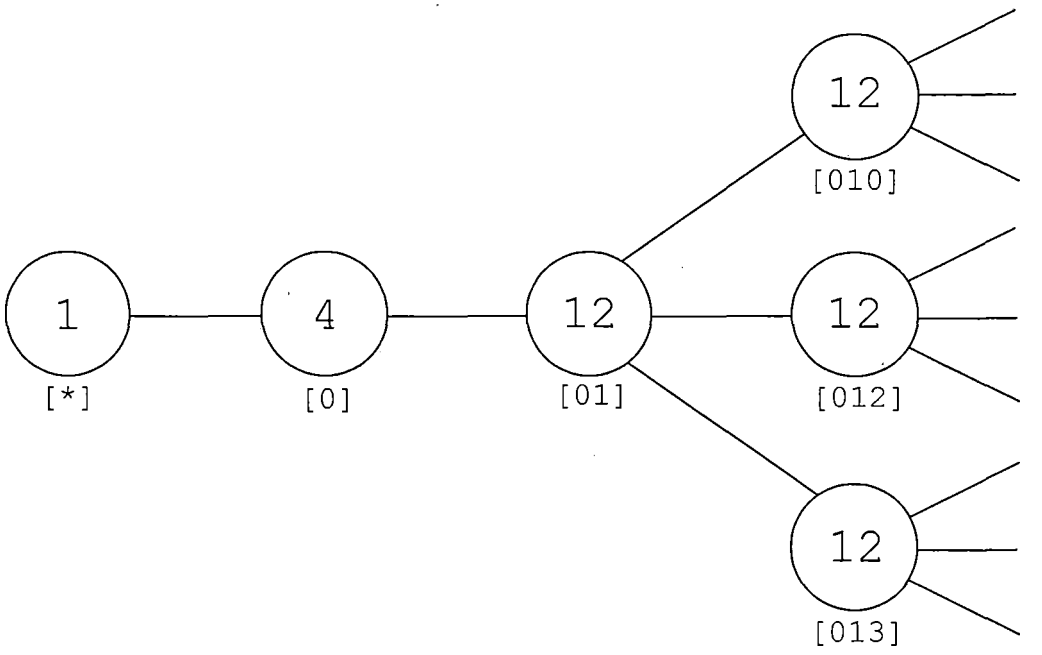
(ii) N permutes $\bar{T}$ transitively.

Conditions (i) and (ii) imply that G is a homomorphic image of $2^{*m} : N$.

N is a group of automorphisms of 2^{*m} which permute by conjugation the m involutory symmetric generators. Hence, the elements of N can be gathered on the left, and every

element of the progenitor can be represented as πw, where $\pi \in N$ and w is a word in the symmetric generators T.

The progenitor $2^{*m} : N$ is an infinite group. For example $2^{*4} : A_4$ is an infinite group as illustrated below



Looking at the double coset Nt_0N we can see that

$Nt_0N = \{Nt_0n \mid n \in N\}$

$= \{Nnn^{-1}t_0n \mid n \in N\}$

$= \{Nnt_0^n \mid n \in N\}$

$= \{Nt_0^n \mid n \in N\}$

$= \{Nt_{n(0)} \mid n \in N\}$

$= \{Nt_0, Nt_1, Nt_2, Nt_3\}$.

In order to obtain a finite homomorphic image of the progenitor, we need to factor by a non-trivial relation of the form πw, where $\pi \in N$ and w is a word in T. We will write presentations for the images of the groups we seek as

$$\frac{2^{*m}:N}{\pi_1 w_1, \ldots, \pi w_r} \cong \langle N, T \mid N_p, t^2, [t, N^0], \pi_1 w_1 = \ldots = \pi w_r \rangle,$$

for r finite number of relations where N_p denotes the presentation for the control group N and [t, N^0] denotes the commutator of t and the stabilizer of 0 in N, N^0.

Furthermore, we are allowing i to stand for the symmetric generator t_i, i.e. 0 denotes the symmetric generator t_0. By a slight abuse of notation we allow i to denote the single coset Nt_i, ij denote Nt_it_j etc., i.e. 01 denotes the single coset Nt_0t_1. Also, we denote the double coset NwN, where w is a word in T, as [w], i.e. [0 1] denotes the double coset Nt_0t_1N. Where appropriate, such as in the labeling of the single cosets of N in G, we allow [w] to denote the single coset Nw, i.e. [0 1] will denote the single coset Nt_0t_1.

To find a non-trivial relation we proceed as follows:

(1) Use Curtis' Lemma (see Curtis [2]) as described below,

or

(2) Look at the conjugacy classes of the control group N. We may use a relation of the form $(\pi w)^n$ and equate it to 1, where $\pi \in N$, w is a word in T, and n is a positive integer.

For example, looking at the progenitor $2^{*4} : A_4$, we may factor by the relation $[(0\ 1\ 2)t_0]^4 = 1$.

Curtis' Lemma: $\langle t_i, t_j \rangle \cap N \leq C_N(N^{ij})$,

where N^{ij} is the point-wise stabilizer of {i, j} in N.

Proof:

Let $n = t_i t_j t_i$, where $n \in \langle t_i, t_j \rangle \cap N$.

Need to show that $n \in C_N(N^{ij})$.

Let $g \in N^{ij}$.

Then $n^g = g^{-1}ng$

$= g^{-1}(t_i t_j t_i)g$ (since $n = t_i t_j t_i$)

$= t_i^g t_j^g t_i^g$

$= t_{g(i)} t_{g(j)} t_{g(i)}$

$= t_i t_j t_i$ (since $g \in N^{ij}$)

$= n$.

And so $n^g = n$

$\Rightarrow$ $g^{-1}ng = n$

$\Rightarrow$ $ng = gn$.

Therefore, $\langle t_i, t_j \rangle \cap N \leq C_N(N^{ij})$.

In like manner, $\langle t_i, t_j, \ldots, t_k \rangle \cap N \leq C_N(N^{ij\cdots k})$, for k finite.

In order to obtain the index of N in G we shall perform a manual double coset enumeration. Thus, we must find all the double cosets [w] and determine how many single cosets each of them contain. We shall know that we have completed the double coset enumeration when the set of right cosets obtained is closed under right multiplication.

So we define

$N^{(w)} = \{\pi \in N \mid Nw\pi = Nw\}$, and

$N^{w} = \{\pi \in N \mid w\pi = \pi w\}$, for w a word in the symmetric generators.

Clearly, $N^w \leq N^{(w)}$. Therefore, factoring the progenitor $2^{*m} : N$ by a non-trivial relation will allow us to find the number of distinct single cosets in the double coset [w] = NwN. This number is given by the index of N over the coset stabilizing group $N^{(w)}$,

$$\text{i.e. } [N : N^{(w)}] = \frac{|N|}{|N^{(w)}|},$$

since

$$N w \pi_1 \neq N w \pi_2 \Leftrightarrow N w \pi_1 \pi_2^{-1} \neq N w$$

$$\Leftrightarrow \pi_1 \pi_2^{-1} \notin N^{(w)}$$

$$\Leftrightarrow N^{(w)}\pi_1 \neq N^{(w)}\pi_2.$$

Moreover, the completion test above is best performed by obtaining the orbits of $N^{(w)}$ on the symmetric generators. We need only identify, for each [w], the double coset to which Nwt_i belongs for one symmetric generator t_i from each orbit.

Now, looking at the progenitor $2^{*5} : A_5$, we have 5 copies of the cyclic group of order 2 extended by A_5, the alternating group on 5 letters.

Example: Consider $G \cong \frac{2^{*5} : A_5}{[(0\ 1\ 2)t_0]^4}$, where $N = A_5$.

This is the progenitor $2^{*5} : A_5$ factored by the relation $[(0\ 1\ 2)t_0]^4$, where (0 1 2) represents the permutaion (5 1 2) of A_5, and t_0 represents the involution t_5.

$N = A_5$ can be generated by:

$x = (1\ 2\ 3)$

$y = (1\ 2\ 4)$

$z = (1\ 2\ 0)$,

where $|x| = |y| = |z| = 3$, and $(x \cdot y)^2 = (x \cdot z)^2 = (y \cdot z)^2 = 1$.

Also, stabilizing 0 in N yields:

$N^0 = \langle (1, 2, 3), (1, 2, 4) \rangle = \langle x, y \rangle$,

and so $[t, N^0] = (t, x), (t, y)$.

The relation $[(0\ 1\ 2)t_0]^4$ may be written as $(z \cdot t)^4$, and so a presentation for the above image may be written as

$$\langle x,y,z,t \mid x^3,\ y^3,\ z^3,\ (x*y)^2,\ (x*z)^2,\ (y*z)^2,\ t^2,\ (t,x),\ (t,y),\ (z*t)^4 \rangle,$$

where $N = A_5 \cong \langle x,y,z \mid x^3,\ y^3,\ z^3,\ (x*y)^2,\ (x*z)^2,\ (y*z)^2 \rangle$.

Expanding the relation $[(0\ 1\ 2)t_0]^4 = 1$ yields:

$(0\ 1\ 2)t_0t_2t_1t_0 = 1$

$\Rightarrow t_0t_1 = (0\ 1\ 2)t_0t_2$

$\Rightarrow \boxed{Nt_0t_1 = Nt_0t_2 \quad (1)}$

Conjugating (1) by $(2\ 3\ 4) \in N$ yields:

$$Nt_0t_1 = Nt_0t_3,$$

and conjugating (1) by $(2\ 4\ 3) \in N$ yields:

$$Nt_0t_1 = Nt_0t_4.$$

Therefore, $Nt_0t_1 = Nt_0t_2 = Nt_0t_3 = Nt_0t_4$.

And so the double coset [0] contains five single cosets, namely, Nt_0, Nt_1, Nt_2, Nt_3, and Nt_4. Meanwhile, the double coset [0 1] has $5 \cdot 4 = 20$ single cosets. But since Nt_0t_1 has four names, then the double coset [0 1] only has $20/4 = 5$ distinct single cosets.

Applying t_2 to (1) yields

$Nt_0t_1t_2 = Nt_0t_2t_2$

$= Nt_0$

$\Rightarrow$ [0 1 2] = [0].

Similarly, [0 1 i] = [0] for all i in {0, 1, 2, 3}.

Applying t_0 to (1) yields:

$Nt_0t_1t_0 = Nt_0t_2t_0$

$\Rightarrow Nt_0t_1t_0 = Nt_0t_2t_0 = Nt_0t_3t_0 = Nt_0t_4t_0$.

But since the coset stabilizing group $N^{(0\ 1\ 0)}$ is transitive on $\{t_0, t_1, t_2, t_3, t_4\}$, then [0 1 0 j] = [0 1] for all j in {0, 1, 2, 3, 4}. And so, the double coset [0 1 0] contains only one single coset. Therefore, we must have found all the double cosets as illustrated in Figure 1.1.

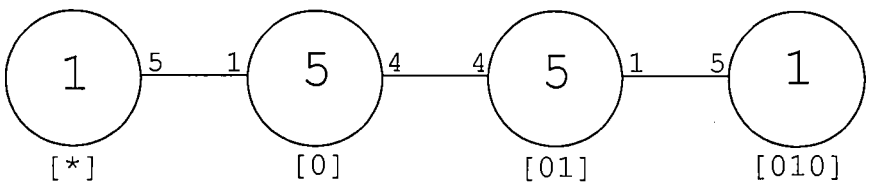


Figure 1.1 Cayley Graph of S_6 over A_5.

From the above Cayley Graph we can see that there are 12 single cosets, each of order 60. Thus, the order of G is given by

$$|G| = (1 + 5 + 5 + 1)60 = 720.$$

Therefore, $G \cong S_6$.

Theorem 1: $G \cong \dfrac{2^{*n} : A_n}{[(0\ 1\ 2)t_0]^4} \cong S_{n+1}$, for $n \geq 3$.

Proof:

Expanding $[(0\ 1\ 2)t_0]^4 = 1$ yields:

$(0\ 1\ 2)t_0t_2t_1t_0 = 1$

$\Rightarrow t_0t_1 = (0\ 1\ 2)t_0t_2$

$\Rightarrow Nt_0t_1 = Nt_0t_2$

$\Rightarrow Nt_0t_1 = Nt_0t_2 = Nt_0t_3 = Nt_0t_4 = \ldots = Nt_0t_{n-1}$.

So the double cosets [0] and [0 1] both contain n single cosets. And [0 1 k] = [0] for all k in {1, 2,. . ., n-1}. But, since the coset stabilizing group $N^{(0\ 1\ 0)}$ is transitive on $\{t_0, t_1, t_2, \ldots, t_{n-1}\}$, then [0 1 0 m] = [0 1] for all m in {0, 1, 2,. . .,n-1}. And so, the double coset [0 1 0] contains only one single coset. Therefore, we must have found all the double cosets as illustrated in Figure 1.2.

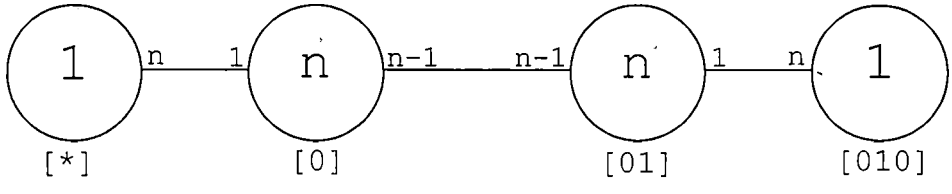


Figure 1.2 Cayley Graph of S_{n+1} over A_n.

From the above Cayley Graph we can see that there are $2(n + 1)$ single cosets, each of order $\frac{n!}{2}$. Thus, the order of G is given by

$$|G| = (1 + n + n + 1)\ \frac{n!}{2} = (n + 1)!.$$

Therefore, $G \cong S_{n+1}$.

Definitions

Below is a list of definitions that may be beneficial to the reader.

G-Set. Let X be a set and G be a group, then X is a G-set if there is a function

$\alpha : G \times X \to X$, denoted by $\alpha(g,x) \mapsto gx$, such that:

(1) $1x = x$, for all $x \in X$, and

(2) $g(hx) = (gh)x$, for all $g,h \in G$ and $x \in X$.

Commutator. If $a, b \in G$, then the commutator of a and b is denoted by $[a,b] = aba^{-1}b^{-1}$.

Complement. Let K be a subgroup of a group G. Then a subgroup $Q \leq G$ is a complement of K in G if

(1) $K \cap Q = 1$, and

(2) $KQ = G$.

Free Group. If X is a set of a group F, then F is a free group with basis X if, for every group G and every function

f: X $\rightarrow$ G, there exists a unique homomorphism φ: F $\rightarrow$ G extending f.

K-Transitive. Let X be a G-set of degree n and let $k \leq n$ be a positive integer, then X is k-transitive if, for every pair of k-tuples having distinct entries in X, say $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$, there is a $g \in G$ such that $gx_i = y_i$ for all i in $\{1, 2, \ldots, k\}$.

Semi-Direct Product. A group G is a semi-direct product of K by Q, if

(1) K is a normal subgroup of G, and

(2) K has a complement $Q_1 \cong Q$.

Simple Group. A group is simple if its only normal subgroups are the group itself and the identity.

Word. A word on X is a sequence $w = (a_1, a_2, \ldots,)$ where $a_i \in X \cup X^{-1} \cup \{1\}$ for all i, such that all $a_i = 1$ from some point on, i.e. there is an integer $n \geq 0$ with $a_i = 1$ for all $i \leq n$. Particularly, the constant sequence $(1, 1, \ldots)$ is a word, denoted by 1, and called the empty word.

CHAPTER TWO

SYMMETRIC REPRESENTAION OF $L_2(11)$ ELEMENTS

Introduction

In this chapter we will show that the group $L_2(11)$ is generated by 4 involutions which are permuted, by conjugation within $L_2(11)$, by a subgroup isomorphic to A_4, and that each element of $L_2(11)$ can be written (not necessarily uniquely) as a permutation of A_4 on 4 letters followed by a word of length less than or equal to four in these generators.

Factoring the progenitor $2^{*4} : A_4$ by the following relations

$[(0\ 1\ 2)t_0]^{11}$, $[(0\ 1)(2\ 3)t_0]^5$, and $[(0\ 1\ 2)t_3t_2t_1t_0]^2$

yields the finite homomorphic image:

$$G \cong \frac{2^{*4} : A_4}{[(0\ 1\ 2)t_0]^{11},\quad [(0\ 1)(2\ 3)t_0]^5,\quad [(0\ 1\ 2)t_3t_2t_1t_0]^2},$$

where the index of A_4 in G equals 55. $G \cong L_2(11)$, the projective special linear group.

A symmetric presentation for the above image is given by:

<x,y,t|x^3, y^3, (x*y)^2, t^2, (t,x), (y*t)^11,

(x*y*t)^5, (y*t^(y^2*x)*t^(y^2)*t^y*t)^2>

where $N = A_4 \cong$ <x,y|x^3, y^3, (x*y)^2>,

and the action of x, y on the symmetric generators is given by

$x \sim (1\ 2\ 3)$

$y \sim (1\ 2\ 0)$.

Double Coset Enumeration

We want to write G as a union of double cosets of the form NxN. In other words, we want to find a set $\{x_1, x_2, x_3, \ldots, x_c\}$, for c finite number of elements of G such that

$$G = Nx_1N \bigcup Nx_2N \bigcup Nx_3N \bigcup \ldots \bigcup Nx_cN, \text{ where } x_i \in G.$$

However, since every element of G can be written as a permutation of N followed by a word w in the symmetric generators, we may write x_i as $x_i = \pi_i w_i$, for $\pi_i \in N$ and w_i is a word in T, where w_1 is the identity.

Therefore, G may be written as

$$\begin{aligned} G &= N\pi_1 w_1 N \bigcup N\pi_2 w_2 N \bigcup N\pi_3 w_3 N \bigcup \ldots \bigcup N\pi_c w_c N \\ &= Nw_1N \bigcup Nw_2N \bigcup Nw_3N \bigcup \ldots \bigcup Nw_cN \\ &= N \bigcup Nw_2N \bigcup Nw_3N \bigcup \ldots \bigcup Nw_cN. \end{aligned}$$

The double coset [0] has 4 distinct single cosets, namely Nt_0, Nt_1, Nt_2, and Nt_3. Meanwhile, the double coset [0 1] contains 12 distinct single cosets.

Applying t_1 to t_0t_1 yields:

$t_0t_1t_1 = t_0$

$\Rightarrow$ [0 1 i] = [0] for i = 1.

Now, expanding the relation $[(0\ 1)(2\ 3)t_0]^5 = 1$ yields:

$(\pi t_0)^5 = 1$, where $\pi = (0\ 1)(2\ 3) \in N$

$\Rightarrow \pi^5(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}(t_0) = 1$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1t_0 = 1$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0 = t_0t_1$

$\Rightarrow Nt_0t_1t_0 = Nt_0t_1$.

Therefore, [0 1 0] = [0 1].

The double coset [0 1 2] contains 12 distinct single cosets.

Now, $Nt_0t_1t_2t_1$ may be written as

$Nt_0t_1t_2t_1 = Nt_0(1\ 2)(0\ 3)t_1t_2$

$= Nt_3t_1t_2$,

but $Nt_3t_1t_2 \in$ [0 1 3] (since $(Nt_3t_1t_2)^{\pi} = Nt_0t_1t_3$, where $\pi = (0\ 2\ 3) \in N$).

Therefore, [0 1 2 1] = [0 1 3].

Now, expanding the relation $[(0\ 1\ 2)t_3t_2t_1t_0]^2 = 1$ yields:

$(\pi t_3t_2t_1t_0)^2 = 1$, where $\pi = (0\ 1\ 2) \in N$

$\Rightarrow \pi^2(t_3t_2t_1t_0)^{\pi}(t_3t_2t_1t_0) = 1$

$\Rightarrow (0\ 1\ 2)t_3t_0t_2t_1t_3t_2t_1t_0 = 1$

$\Rightarrow \boxed{t_0t_1t_2t_3 = (0\ 2\ 1)t_3t_0t_2t_1 \quad (1)}$

$\Rightarrow [t_0t_1t_2t_3]^{(0\ 1\ 3)} = [(0\ 2\ 1)t_3t_0t_2t_1]^{(0\ 1\ 3)}$

$\Rightarrow \boxed{t_1t_3t_2t_0 = (1\ 2\ 3)t_0t_1t_2t_3 . \quad (2)}$

Therefore, $Nt_0t_1t_2t_3 = Nt_3t_0t_2t_1 = Nt_1t_3t_2t_0$.

Therefore, the double coset [0 1 2 3] has 4 distinct single cosets, namely $Nt_0t_1t_2t_3$, $Nt_0t_2t_3t_1$, $Nt_0t_3t_1t_2$, and $Nt_1t_2t_0t_3$, each with three names.

Now, applying t_1 to (1) yields:

$t_0t_1t_2t_3t_1 = (0\ 2\ 1)t_3t_0t_2t_1t_1$

$= (0\ 2\ 1)t_3t_0t_2$

but $t_3t_0t_2 \in$ [0 1 2] (since $(t_3t_0t_2)^{\pi} = t_0t_1t_2$, where $\pi = (3\ 0\ 2) \in N$).

Therefore, [0 1 2 3 1] = [0 1 2].

However, since $N^{(0\ 1\ 2\ 3)}$ has orbits {2} and {0, 1, 3} then [0 1 2 3 j] = [0 1 2] for all j in {0, 1, 3}.

Now, $Nt_0t_1t_2t_3t_2$ may be written as

$Nt_0t_1t_2t_3t_2 = Nt_0t_1(3\ 2)(1\ 0)t_2t_3$

$= Nt_1t_0t_2t_3$,

but $Nt_1t_0t_2t_3 \in$ [0 1 3 2] (since $(Nt_1t_0t_2t_3)^{\pi} = Nt_0t_1t_3t_2$, where $\pi = (0\ 1)(2\ 3) \in N$).

Therefore, [0 1 2 3 2] = [0 1 3 2].

From (2) we have

$t_1t_3t_2t_0 = (1\ 2\ 3)t_0t_1t_2t_3$

$\Rightarrow (1\ 3\ 2)t_3t_0t_1t_3 = t_2t_3t_0t_2$

and

$t_1t_3t_2t_0 = (1\ 2\ 3)t_0t_1t_2t_3$

$\Rightarrow \boxed{t_1t_2t_3t_1 = (0\ 2\ 1)t_2t_3t_0t_2 \quad (3)}$

$\Rightarrow [t_1t_2t_3t_1]^{(0\ 2)(1\ 3)} = [(0\ 2\ 1)t_2t_3t_0t_2]^{(0\ 2)(1\ 3)}$

$\Rightarrow t_3t_0t_1t_3 = (2\ 0\ 3)t_0t_1t_2t_0$

$\Rightarrow (1\ 2\ 3)t_2t_3t_0t_2 = (2\ 0\ 3)t_0t_1t_2t_0$

$\Rightarrow (1\ 2)(0\ 3)t_2t_3t_0t_2 = t_0t_1t_2t_0$

Therefore, $Nt_0t_1t_2t_0 = Nt_2t_3t_0t_2$.

The double coset [0 1 2 0] contains $4 \cdot 3 \cdot 2 \cdot 1 = 12$ single cosets, but since $Nt_0t_1t_2t_0$ has two names, then the number of distinct single cosets in [0 1 2 0] $= \frac{12}{2} = 6$.

So, applying t_2 to $Nt_0t_1t_2t_0 = Nt_2t_3t_0t_2$ yields:

$Nt_0t_1t_2t_0t_2 = Nt_2t_3t_0t_2t_2$

$= Nt_2t_3t_0,$

but $Nt_2t_3t_0 \in [0\ 1\ 2]$ (since $(Nt_2t_3t_0)^{\pi} = Nt_0t_1t_2$, where $\pi = (0\ 2)(1\ 3) \in N$).

Therefore, [0 1 2 0 2] = [0 1 2].

However, since $N^{(0\ 1\ 2\ 0)}$ has orbits {0, 2} and {1, 3} then [0 1 2 0 k] = [0 1 2] for all k in {0, 2}.

Now, $t_0t_1t_2t_0t_3$ may be written as

$t_0t_1t_2t_0t_3 = t_0(0\ 2\ 3)t_2t_3t_0t_1$

$= (0\ 2\ 3)t_2t_2t_3t_0t_1$

$= (0\ 2\ 3)t_3t_0t_1,$

but $t_3t_0t_1 \in [0\ 1\ 3]$ (since $(t_3t_0t_1)^{\pi} = t_0t_1t_3,$

where $\pi = (0\ 1\ 3) \in N$).

Therefore, $[0\ 1\ 2\ 0\ 3] = [0\ 1\ 3]$, which implies that

$[0\ 1\ 2\ 0\ h] = [0\ 1\ 3]$ for all h in {1, 3}.

The double coset $[0\ 1\ 3]$ contains 12 distinct single cosets.

Now, $Nt_0t_1t_3t_1$ may be written as

$Nt_0t_1t_3t_1 = Nt_0(1\ 3)(0\ 2)t_1t_3$

$= Nt_2t_1t_3,$

but $Nt_2t_1t_3 \in [0\ 1\ 2]$ (since $(Nt_2t_1t_3)^{\pi} = Nt_0t_1t_2,$

where $\pi = (0\ 3\ 2) \in N$).

Therefore, $[0\ 1\ 3\ 1] = [0\ 1\ 2]$.

Conjugating (3) by $(1\ 3)(2\ 0)$ yields:

$[t_1t_2t_3t_1]^{(1\ 3)(0\ 2)} = [(0\ 2\ 1)t_2t_3t_0t_2]^{(1\ 3)(0\ 2)}$

$\Rightarrow t_3t_0t_1t_3 = (2\ 0\ 3)t_0t_1t_2t_0,$

but $t_3t_0t_1t_3 \in [0\ 1\ 3\ 0]$ (since $(t_3t_0t_1t_3)^{\pi} = Nt_0t_1t_3t_0,$

where $\pi = (0\ 1\ 3) \in N$).

Therefore, $[0\ 1\ 3\ 0] = [0\ 1\ 2\ 0]$.

From (2) we have

$t_1t_3t_2t_0 = (1\ 2\ 3)t_0t_1t_2t_3$

$\Rightarrow t_0(1\ 3\ 2)t_1t_3t_2 = t_1t_2t_3t_0$

$\Rightarrow Nt_0t_1t_3t_2 = Nt_1t_2t_3t_0$

$\Rightarrow (Nt_0t_1t_3t_2)^{(0\ 1\ 2)} = (Nt_1t_2t_3t_0)^{(0\ 1\ 2)}$

$\Rightarrow Nt_1t_2t_3t_0 = Nt_2t_0t_3t_1.$

Therefore, $Nt_0t_1t_3t_2 = Nt_1t_2t_3t_0 = Nt_2t_0t_3t_1.$

Therefore, the double coset [0 1 3 2] has 4 distinct single cosets, namely $Nt_0t_1t_3t_2$, $Nt_0t_2t_1t_3$, $Nt_0t_3t_2t_1$, and $Nt_1t_3t_0t_2$, each with three names.

Applying t_0 to $Nt_0t_1t_3t_2 = Nt_1t_2t_3t_0$ yields:

$Nt_0t_1t_3t_2t_0 = Nt_1t_2t_3t_0t_0$

$= Nt_1t_2t_3,$

but $Nt_1t_2t_3 \in$ [0 1 3] (since $(Nt_1t_2t_3)^{\pi} = Nt_0t_1t_3$, where $\pi = (0\ 2\ 1) \in N$).

Therefore, [0 1 3 2 0] = [0 1 3].

However, since $N^{(0\ 1\ 3\ 2)}$ has orbits {3} and {0, 1, 2} then [0 1 3 2 m] = [0 1 3] for all m in {0, 1, 2}.

Now, $Nt_0t_1t_3t_2t_3$ may be written as

$Nt_0t_1t_3t_2t_3 = Nt_0t_1(2\ 3)(0\ 1)t_3t_2$

$= Nt_1t_0t_3t_2,$

but $Nt_1t_0t_3t_2 \in$ [0 1 2 3] (since $(Nt_1t_0t_3t_2)^{\pi} = Nt_0t_1t_2t_3$,

where $\pi = (0\ 1)(2\ 3) \in N$).

Therefore, [0 1 3 2 3] = [0 1 2 3].

The set of all double cosets [w] = NwN, coset stabilizing subgroups $N^{(w)}$, and the number of single cosets are exhibited in Table 2.1. The table shows that the Cayley Graph of G over N has the form as displayed in Figure 2.1.

Table 2.1 The Double Cosets [w] = NwN in G, Where N = A_4

NwN	Reason	Number of single cosets
[*]	N is transitive	1
[0]	$N^{(0)}$ = <(123)> has orbits {0}, {1, 2, 3}	4
[00]=[*]		
[01]	$N^{(0\ 1)}$ = <*e*> has orbits {0}, {1}, {2}, {3} 01 = (01)(23)010	12
[010]=[01]		
[011]=[0]		
[012]	$N^{(0\ 1\ 2)}$ = <*e*> has orbits {0}, {1}, {2}, {3} 1320 = (123)0123 ⇒ (132)3013 = 2302, and 1320 = (123)0123 ⇒ 1231 = (021)2302 ⇒ 0120 = (12)(03)2302.	12
[0120]	$N^{(0\ 1\ 2\ 0)}$ = <(02)(13)> has orbits {0,2}, {1,3} and for all i in {0,2}	6
[0120i]=[012]		
	01203 = 0(023)2301 = (023)301 and $(301)^{\pi}$ = 013 where $\pi = (301) \in N$	
[01203]=[013]		
	0121 = (12)(03)312 and $(312)^{\pi}$ = 013, where	

```
          π = (230) ∈ N
[0121] = [013]

[0122] = [01]

[0123]    N^(0 1 2 3) = <(031)> has orbits {2}, {0,1,3}        4
          and 0123 ~ 3021 ~ 1320 and for all j in
          {0, 1, 3}
[0123j] = [012]

          01232 = 01(32)(10)23 = (32)(10)1023, and
          (1023)^π = 0132, where π = (01)(23) ∈ N
[01232] = [0132]
[013]     N^(0 1 3) = <e> has orbits {0}, {1}, {2}, {3}       12

[0133] = [01]
          0130 = (032)2012 and (2012)^π = 0120 where
          π = (201) ∈ N
[0130]=[0120]

          0131 = 0(31)(02)13 = (31)(02)213 and (213)^π
          = 012 where π = (320) ∈ N
[0131]=[012]

[0132]    N^(0 1 3 2) = <(102)> has orbits {3}, {1,2,0}        4
          0132 ~ 2031 ~ 1230 and for all k in {1,2,0}
[0132k]=[013]

          01323 = 01(23)(01)32 = (23)(01)1032 and
          (1032)^π = 0123 where π = (01)(23) ∈ N
[01323]=[0123]
```

The Cayley graph of $L_2(11)$ over A_4 has as vertices the set of right cosets of N in G, i.e. the set Nw_i, where w_i are words in T. The diagram below illustrates each orbit of N in its action on the vertices by right multiplication which is represented by a single node, labeled with the number of

vertices that it contains. Lines in between these nodes are labeled with integers to denote how many edges from a vertex of one node will lead to the vertices of the other.

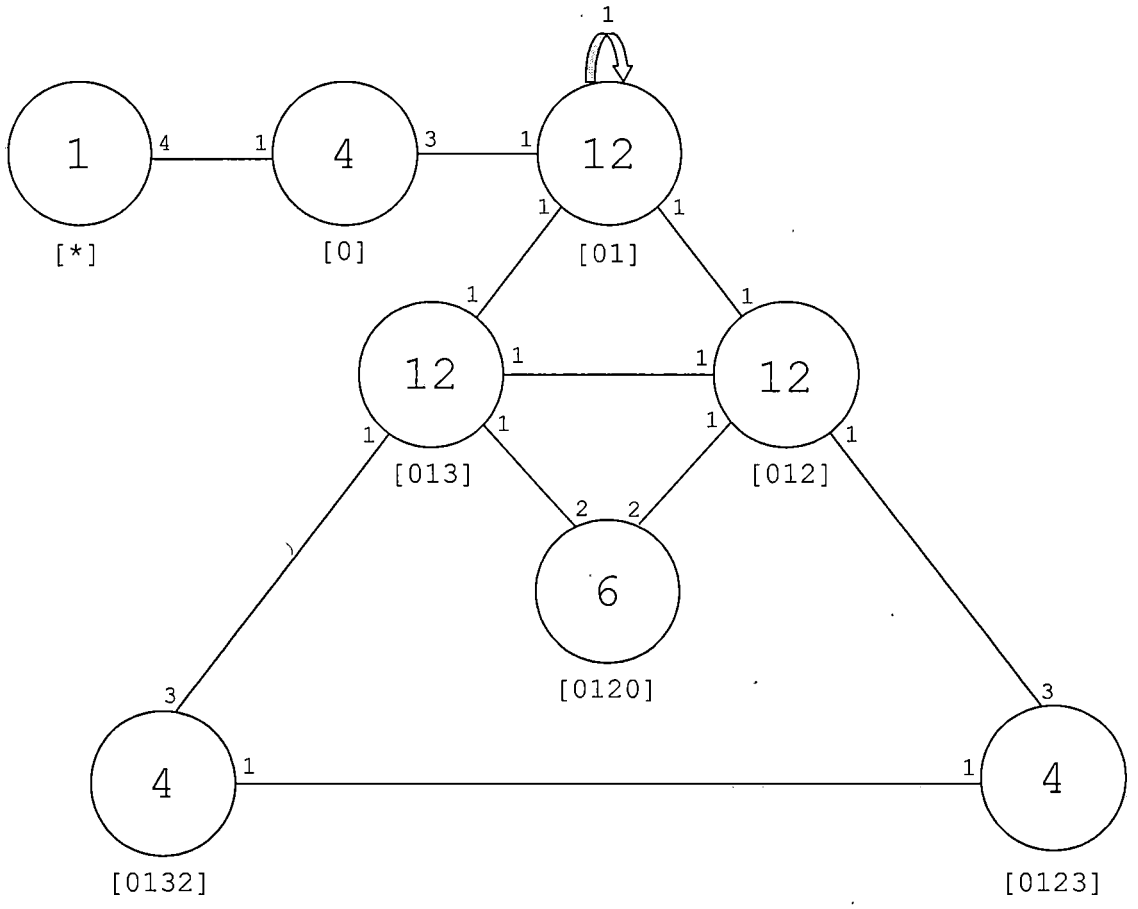


Figure 2.1 Cayley Graph of $L_2(11)$ over A_4

At this point we can calculate the action of the four symmetric generators on the cosets of $L_2(11)$ over A_4. For example, in order to calculate this for the symmetric generator t_0, we start with the identity coset N or * and

then multiply on the right by t_0, in which the result is Nt_0 or 0. We repeat this process again by multiplying on the right hand side once more by t_0, the result is $Nt_0t_0 = N$. We then start with a new single coset and repeat the process in like manner. It should be mentioned that since the symmetric generators are involutions, then each symmetric generator will be expressed as a product of two-cycles.

The action of the symmetric generators on the cosets of N in G is given by:

t_0: (* 0)(1 10)(2 20)(3 30)(01 010)(02 020)(03 030)
(12 120)(13 130)(21 210)(23 230)(31 310)(32 320)
(012 0120)(023 0230)(031 0310)(103 1030)(132 1320)
(201 2010)(213 2130)(302 3020)(321 3210)(013 0130)
(021 0210)(032 0320)(123 1230)(231 2310)(312 3120)
(1203 12030)

t_1: (* 1)(0 01)(2 21)(3 31)(02 021)(03 031)(10 101)(12 121)
(13 131)(20 201)(23 231)(30 301)(32 321)(012 0121)
(023 0231)(103 1031)(120 1201)(132 1321)(213 2131)
(230 2301)(302 3021)(310 3101)(032 0321)(102 1021)
(123 1231)(130 1301)(203 2031)(320 3201)(0312 03121)

t_2: (* 2)(0 02)(1 12)(01 012)(03 032)(10 102)(13 132)
(20 202)(21 212)(23 232)(30 302)(31 312)(023 0232)
(031 0312)(103 1032)(120 1202)(201 2012)(213 2132)
(230 2302)(310 3102)(321 3212)(013 0132)(130 1302)
(203 2032)(210 2102)(231 2312)(301 3012)(0123 01232)

t_3: (* 3)(0 03)(1 13)(2 23)(01 013)(02 023)(10 103)(12 123)
(20 203)(21 213)(30 303)(31 313)(32 323)(012 0123)
(031 0313)(120 1203)(132 1323)(201 2013)(230 2303)
(302 3023)(310 3103)(321 3213)(021 0213)(102 1023)
(210 2103)(301 3013)(312 3123)(320 3203)(0231 02313).

Now, we label the single cosets of N in G as follows:

Label	Single Coset
55.	[N]
1.	[1]
2.	[2]
3.	[3]
4.	[0]
5.	[0 1]
6.	[0 2]
7.	[0 3]
8.	[1 0]
9.	[1 2]
10.	[1 3]
11.	[2 0]
12.	[2 1]
13.	[2 3]
14.	[3 0]
15.	[3 1]
16.	[3 2]
17.	[0 1 2]
18.	[0 2 3]
19.	[0 3 1]
20.	[1 0 3]
21.	[1 2 0]
22.	[1 3 2]
23.	[2 0 1]
24.	[2 1 3]
25.	[2 3 0]
26.	[3 0 2]
27.	[3 1 0]
28.	[3 2 1]
29.	[0 1 3]
30.	[0 2 1]
31.	[0 3 2]
32.	[1 0 2]
33.	[1 2 3]
34.	[1 3 0]
35.	[2 0 3]
36.	[2 1 0]
37.	[2 3 1]
38.	[3 0 1]
39.	[3 1 2]
40.	[3 2 0]
41.	[0 1 2 0]
42.	[0 2 3 0]
43.	[0 3 1 0]
44.	[1 0 3 1]
45.	[1 3 2 1]
46.	[2 1 3 2]
47.	[0 1 2 3]
48.	[0 2 3 1]
49.	[0 3 1 2]
50.	[1 2 0 3]
51.	[0 1 3 2]
52.	[0 2 1 3]
53.	[0 3 2 1]
54.	[1 3 0 2]

Using the above labeling the four symmetric generators can be re-written as:

t_0 = (55,4)(1,8)(2,11)(3,14)(9,21)(10,34)(12,36)(13,25)
(15,27)(16,40)(17,41)(18,42)(19,43)(20,35)(22,47)
(23,38)(24,48)(26,32)(28,49)(29,45)(30,46)(31,44)
(33,51)(37,52)(39,53)(50,54)

t_1 = (55,1)(4,5)(2,12)(3,15)(6,30)(7,19)(11,23)(13,37)(14,
38)(16,28)(17,39)(18,48)(20,44)(21,43)(22,45)(24,29)

(25,50)(26,47)(27,36)(31,53)(32,42)(33,41)(34,46)(35,51)(40,54)(49,52)

t_2 = (55,2)(4,6)(1,9)(3,16)(5,17)(7,31)(8,32)(10,22)(14,26)(15,39)(18,33)(19,49)(20,48)(21,40)(23,45)(24,46)(25,41)(27,50)(28,30)(29,51)(34,54)(35,43)(36,44)(37,42)(38,52)(47,53)

t_3 = (55,3)(4,7)(1,10)(2,13)(5,29)(6,18)(8,20)(9,33)(11,35)(12,24)(17,47)(19,37)(21,50)(22,31)(23,49)(25,34)(26,46)(27,42)(28,44)(30,52)(32,53)(36,54)(38,41)(39,43)(40,45)(48,51).

Representation of Elements of $L_2(11)$

Every element of $L_2(11)$ can be represented as πw, where π is a permutation of A_4, on 4 letters, and w is a product of at most four symmetric generators.

In general, any element of G can be written as πw, where $\pi \in N$ and w is a word in T. We refer to this as a symmetric representation of an element of the group. An algorithm to find such is given below.

Algorithm

Let $\alpha \in G$,

then $N^{\alpha} = N\alpha$ (since the action is right multiplication)

$\Rightarrow N\alpha = Nt_it_j$ (since $G = \langle t_1, t_2, \ldots, t_n \rangle$)

$\Rightarrow N\alpha t_jt_i = N$

$\Rightarrow \alpha t_jt_i = n$, for $n \in N$

$\Rightarrow \alpha = nt_it_j$, a symmetric representation for α.

Next, we find the action of n as described above on the set of symmetric generators $\{t_1, t_2, \ldots, t_n\}$. And so, we have a symmetric representation for α.

For example:

Let α = (1, 40, 35)(2, 33, 54)(3, 30, 32)(4, 53, 38)(5, 11, 18)(6, 13, 23)(7, 12, 44)(8, 21, 46) (9, 24, 20) (10, 42, 17)(14, 28, 19)(15, 25, 45)(16, 43, 26) (22, 27, 41)(29, 39, 52)(34, 37, 51)(47, 50, 55),

for $\alpha \in L_2(11)$, where α is a permutation on 55 letters.

Now $N^{\alpha} = N\alpha$

$\Rightarrow$ $55^{\alpha} = 47$ (Since N is labeled as 55 and 55 goes to 47 under α)

$\Rightarrow$ $N\alpha = Nt_0t_1t_2t_3$ (Since $Nt_0t_1t_2t_3$ is labeled as 47)

$\Rightarrow$ $N\alpha t_3t_2t_1t_0 = N$

$\Rightarrow$ $\alpha t_3t_2t_1t_0 = n$, for $n \in A_4$

$\Rightarrow$ $\alpha = nt_0t_1t_2t_3$, a symmetric representation for α.

Now, what is n?

$\alpha t_3t_2t_1t_0$ = (1, 2, 4)(5, 9, 11)(6, 8, 12)(7, 10, 13)(14, 15, 16)(17, 21, 23)(18, 20, 24)(19, 22, 25) (26, 27, 28)(29, 33, 35)(30, 32, 36)(31, 34, 37)(38, 39, 40)(41, 43, 45)(42, 44, 46)(47, 50, 49)(52, 53, 54)

Therefore, the action of $\alpha t_3t_2t_1t_0$ on the symmetric generators is given by:

$Nt_1 = 1 \rightarrow 2 = Nt_2$

$Nt_2 = 2 \rightarrow 4 = Nt_0$

$Nt_3 = 3 \rightarrow 3 = Nt_3$

$Nt_0 = 4 \rightarrow 1 = Nt_1$

$\Rightarrow n = (1\ 2\ 0)$.

Therefore, a symmetric representation for α is given by $\alpha = (1\ 2\ 0)t_0t_1t_2t_3$, where $(1\ 2\ 0) \in A_4$.

Permutation Representation of an Element Whose Symmetric Representation Is Given

Given a symmetric representation for any element of $L_2(11)$, we can easily recover its permutation representation on 55 lettters. An algorithm is given below.

Algorithm

Given a symmetric representation πw for any element $\beta \in G$, we need to find the action of π on each of the single cosets of N in G, in which we will denote the result by n. So a permutation representation for β will be given by the product of n and the symmetric generators as given in the symmetric representation.

For example:

Let β = (2 3)t_1t_2 be the symmetric representation of the element $\beta \in G$. Then the action of π = (2 3) $\in$ N on each of the fifty-five single cosets of N = A_4 in $L_2(11)$ is given by:

Label	Single Coset	action under π	Coset	Single Coset
55.	[N]	------>	[N]	= 55
1.	[1]	------>	[1]	= 1
2.	[2]	------>	[3]	= 3
3.	[3]	------>	[2]	= 2
4.	[0]	------>	[0]	= 4
5.	[0 1]	------>	[0 1]	= 5
6.	[0 2]	------>	[0 3]	= 7
7.	[0 3]	------>	[0 2]	= 6
8.	[1 0]	------>	[1 0]	= 8
9.	[1 2]	------>	[1 3]	= 10
10.	[1 3]	------>	[1 2]	= 9
11.	[2 0]	------>	[3 0]	= 14
12.	[2 1]	------>	[3 1]	= 15
13.	[2 3]	------>	[3 2]	= 16
14.	[3 0]	------>	[2 0]	= 11
15.	[3 1]	------>	[2 1]	= 12
16.	[3 2]	------>	[2 3]	= 13
17.	[0 1 2]	------>	[0 1 3]	= 29
18.	[0 2 3]	------>	[0 3 2]	= 31
19.	[0 3 1]	------>	[0 2 1]	= 30
20.	[1 0 3]	------>	[1 0 2]	= 32
21.	[1 2 0]	------>	[1 3 0]	= 34
22.	[1 3 2]	------>	[1 2 3]	= 33
23.	[2 0 1]	------>	[3 0 1]	= 38
24.	[2 1 3]	------>	[3 1 2]	= 39
25.	[2 3 0]	------>	[3 2 0]	= 40
26.	[3 0 2]	------>	[2 0 3]	= 35
27.	[3 1 0]	------>	[2 1 0]	= 36
28.	[3 2 1]	------>	[2 3 1]	= 37
29.	[0 1 3]	------>	[0 1 2]	= 17
30.	[0 2 1]	------>	[0 3 1]	= 19
31.	[0 3 2]	------>	[0 2 3]	= 18

32. [1 0 2] ------> [1 0 3] = 20
33. [1 2 3] ------> [1 3 2] = 22
34. [1 3 0] ------> [1 2 0] = 21
35. [2 0 3] ------> [3 0 2] = 26
36. [2 1 0] ------> [3 1 0] = 27
37. [2 3 1] ------> [3 2 1] = 28
38. [3 0 1] ------> [2 0 1] = 23
39. [3 1 2] ------> [2 1 3] = 24
40. [3 2 0] ------> [2 3 0] = 25
41. [0 1 2 0]------> [0 1 3 0] = 45
42. [0 2 3 0]------> [0 3 2 0] = 44
43. [0 3 1 0]------> [0 2 1 0] = 46
44. [1 0 3 1]------> [1 0 2 1] = 42
45. [1 3 2 1]------> [1 2 3 1] = 41
46. [2 1 3 2]------> [3 1 2 3] = 43
47. [0 1 2 3]------> [0 1 3 2] = 51
48. [0 2 3 1]------> [0 3 2 1] = 53
49. [0 3 1 2]------> [0 2 1 3] = 52
50. [1 2 0 3]------> [1 3 0 2] = 54
51. [0 1 3 2]------> [0 1 2 3] = 47
52. [0 2 1 3]------> [0 3 1 2] = 49
53. [0 3 2 1]------> [0 2 3 1] = 48
54. [1 3 0 2]------> [1 2 0 3] = 50

$\Rightarrow$ n = (2, 3)(6, 7)(9, 10)(11, 14)(12, 15)(13, 16)(17, 29)
(18, 31)(19, 30)(20, 32)(21, 34)(22, 33)(23, 38)(24,
39)(25, 40)(26, 35)(27, 36)(28, 37)(41, 45)(42, 44)
(43, 46)(47, 51)(48, 53)(49, 52)(50, 54).

So, a permutation representation for β is given by

β = (2, 3)(6, 7)(9, 10)(11, 14)(12, 15)(13, 16)(17, 29)(18,
31)(19, 30)(20, 32)(21, 34)(22, 33)(23, 38)(24, 39)
(25, 40)(26, 35)(27, 36)(28, 37)(41, 45)(42, 44)(43,
46)(47, 51)(48, 53)(49, 52)(50, 54)t_1t_2
= (2, 3)(6, 7)(9, 10)(11, 14)(12, 15)(13, 16)(17, 29)

(18, 31)(19, 30)(20, 32)(21, 34)(22, 33)(23, 38)(24, 39)(25, 40)(26, 35)(27, 36)(28, 37)(41, 45)(42, 44)(43, 46)(47, 51)(48, 53)(49, 52)(50, 54)(1, 55)(2, 12)(3, 15)(4, 5)(6, 30)(7, 19)(11, 23)(13, 37)(14, 38)(16, 28)(17, 39)(18, 48)(20, 44)(21, 43)(22, 45)(24, 29)(25, 50)(26, 47)(27, 36)(31, 53)(32, 42)(33, 41)(34, 46)(35, 51)(40, 54)(49, 52) (1, 9)(2, 55)(3, 16)(4, 6)(5, 17)(7, 31)(8, 32)(10, 22)(14, 26)(15, 39)(18, 33)(19, 49)(20, 48)(21, 40)(23, 45)(24, 46)(25, 41)(27, 50)(28, 30)(29, 51)(34, 54)(35, 43)(36, 44)(37, 42)(38, 52)(47, 53)

$\Rightarrow \beta$ = (1, 2, 39, 51, 14, 45, 18, 47, 43, 54, 41, 10)(3, 12, 16, 42, 48, 7, 28, 13, 30, 31, 20, 37)(4, 17, 46, 40, 27, 50, 21, 24, 5, 6, 49, 19)(8, 32, 36, 44)(9, 22, 25, 34, 35, 53, 33, 23, 26, 29, 15, 55)(11, 52, 38).

Proof of Isomorphism

Double coset enumeration as displayed in Figure 2.1 shows that the group defined by the symmetric presentation contains a homomorphic image of $N \cong A_4$ to index at most

$$\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(0\ 1)}|} + \frac{|N|}{|N^{(0\ 1\ 2)}|} + \frac{|N|}{|N^{(0\ 1\ 3)}|} +$$

$$\frac{|N|}{|N^{(0\ 1\ 2\ 3)}|} + \frac{|N|}{|N^{(0\ 1\ 3\ 2)}|} + \frac{|N|}{|N^{(0\ 1\ 2\ 0)}|}$$

$$= \frac{12}{12} + \frac{12}{3} + \frac{12}{1} + \frac{12}{1} + \frac{12}{1} + \frac{12}{3} + \frac{12}{3} + \frac{12}{2}$$

$$= 1 + 4 + 12 + 12 + 12 + 4 + 4 + 6$$

$$= 55.$$

As a result, the index of N in G is at most 55.

Therefore, the order of the image group G is at most

$$55 \cdot |N| = 55 \cdot 12 = 660.$$

Moreover, the action by right multiplication on the 55 cosets is implicit in the enumeration, and we can easily verify that the image is the projective special linear group $L_2(11)$.

Furthermore, the order of G can by confirmed by regarding G as a permutation group on the 55 cosets that we have found. The action of x and y on the single cosets of N in G is given by:

x: (1, 2, 3)(5, 6, 7)(8, 11, 14)(9, 13, 15)(10, 12, 16)
(17, 18, 19)(20, 23, 26)(21, 25, 27)(22, 24, 28)(29,
30, 31)(32, 35, 38)(33, 37, 39)(34, 36, 40)(41, 42,
43)(44, 45, 46)(47, 48, 49)(51, 52, 53)

y: (1, 2, 4)(5, 9, 11)(6, 8, 12)(7, 10, 13)(14, 15, 16)
(17, 21, 23)(18, 20, 24)(19, 22, 25)(26, 27, 28)(29,

33, 35)(30, 32, 36)(31, 34, 37)(38, 39, 40)(41, 43, 45)
(42, 44, 46)(47, 50, 49)(52, 53, 54).

As a result, the action of x and y on $\{t_0, t_1, t_2, t_3\}$ is given by:

x: (t_1, t_2, t_3)

y: (t_1, t_2, t_0).

We note that the order of the product $x \cdot y$ equals 2, and so $N = \langle x, y \rangle \cong A_4$.

The action of the symmetric generator t_0 on the single cosets of N in G is given by:

t_0: (72, 7)(1, 8)(2, 9)(3, 10)(4, 11)(5, 12)(6, 13)(14, 31)
(15, 50)(16, 33)(17, 34)(18, 30)(19, 36)(20, 40)(21,
29)(22, 55)(23, 38)(24, 45)(25, 47)(26, 54)(27, 48)(28,
41)(32, 67)(35, 68)(37, 58)(39, 61)(42, 64)(43, 69)(44,
66)(46, 62)(49, 70)(51, 63)(52, 60)(53, 59)(56, 57)(65,
71)

Now, we need to verify that our relations hold true in $L_2(11)$.

Looking at our first relation we can see that

$t_0^{(t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0)}$ = (1, 55)(2, 12)(3, 15)(4, 5)(6, 30)(7, 19)
(11, 23)(13, 37)(14, 38)(16, 28)(17, 39)
(18, 48)(20, 44)(21, 43)(22, 45)(24, 29)

$$(25, 50)(26, 47)(27, 36)(31, 53)(32, 42)$$

$$(33, 41)(34, 46)(35, 51)(40, 54)(49, 52)$$

$$= t_1$$

$$t_1^{(t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0)} = (1, 9)(2, 55)(3, 16)(4, 6)(5, 17)(7, 31)$$

$$(8, 32)(10, 22)(14, 26)(15, 39)(18, 33)$$

$$(19, 49)(20, 48)(21, 40)(23, 45)(24, 46)$$

$$(25, 41)(27, 50)(28, 30)(29, 51)(34, 54)$$

$$(35, 43)(36, 44)(37, 42)(38, 52)(47, 53)$$

$$= t_2$$

$$t_2^{(t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0)} = (1, 8)(2, 11)(3, 14)(4, 55)(9, 21)(10,$$

$$34)(12, 36)(13, 25)(15, 27)(16, 40)(17,$$

$$41)(18, 42)(19, 43)(20, 35)(22, 47)(23,$$

$$38)(24, 48)(26, 32)(28, 49)(29, 45)(30,$$

$$46)(31, 44)(33, 51)(37, 52)(39, 53)(50,$$

$$54)$$

$$= t_0$$

$$t_3^{(t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0)} = (1, 10)(2, 13)(3, 55)(4, 7)(5, 29)(6, 18)$$

$$(8, 20)(9, 33)(11, 35)(12, 24)(17, 47)$$

$$(19, 37)(21, 50)(22, 31)(23, 49)(25, 34)$$

$$(26, 46)(27, 42)(28, 44)(30, 52)(32, 53)$$

$$(36, 54)(38, 41)(39, 43)(40, 45)(48, 51)$$

$$= t_3.$$

Thus, $t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0$ acts as the permutation (0 1 2) on the symmetric generators, that is

$$t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0 = (0\ 1\ 2)$$

$$\Rightarrow t_0t_1t_2t_0t_1t_2 = (0\ 2\ 1)t_1t_0t_2t_1t_0.$$

Therefore, our first relation holds true in $L_2(11)$.

Now, looking at our second relation, we can see that

$t_0^{(t_0t_1t_0t_1t_0)}$ = (1, 55)(2, 12)(3, 15)(4, 5)(6, 30)(7, 19)(11, 23)(13, 37)(14, 38)(16, 28)(17, 39)(18, 48)(20, 44)(21, 43)(22, 45)(24, 29)(25, 50)(26, 47)(27, 36)(31, 53)(32, 42)(33, 41)(34, 46)(35, 51)(40, 54)(49, 52)

= t_1

$t_1^{(t_0t_1t_0t_1t_0)}$ = (1, 8)(2, 11)(3, 14)(4, 55)(9, 21)(10, 34)(12, 36)(13, 25)(15, 27)(16, 40)(17, 41)(18, 42)(19, 43)(20, 35)(22, 47)(23, 38)(24, 48)(26, 32)(28, 49)(29, 45)(30, 46)(31, 44)(33, 51)(37, 52)(39, 53)(50, 54)

= t_0

$t_2^{(t_0t_1t_0t_1t_0)}$ = (1, 10)(2, 13)(3, 55)(4, 7)(5, 29)(6, 18)(8, 20)(9, 33)(11, 35)(12, 24)(17, 47)(19, 37)(21, 50)(22, 31)(23, 49)(25, 34)(26, 46)(27, 42)(28, 44)(30, 52)(32, 53)(36, 54)(38, 41)(39, 43)(40, 45)

(48, 51)

$= t_3$

$t_3^{(t_0t_1t_0t_1t_0)}$ = (1, 9)(2, 55)(3, 16)(4, 6)(5, 17)(7, 31)(8, 32)

(10, 22)(14, 26)(15, 39)(18, 33)(19, 49)(20, 48)

(21, 40)(23, 45)(24, 46)(25, 41)(27, 50)(28, 30)

(29, 51)(34, 54)(35, 43)(36, 44)(37, 42)(38, 52)

(47, 53)

$= t_2$.

Thus, $t_0t_1t_0t_1t_0$ acts as the permutation (0 1)(2 3) on the symmetric generators, that is

$$t_0t_1t_0t_1t_0 = (0\ 1)(2\ 3)$$

$$\Rightarrow t_0t_1t_0 = (0\ 1)(2\ 3)t_0t_1.$$

Therefore, our second relation holds true in $L_2(11)$.

Now, looking at our third relation we can see that:

$t_0^{(t_3t_0t_2t_1t_3t_2t_1t_0)}$ = (1, 55)(2, 12)(3, 15)(4, 5)(6, 30)(7, 19)(11,

23)(13, 37)(14, 38)(16, 28)(17, 39)(18, 48)

(20, 44)(21, 43)(22, 45)(24, 29)(25, 50)(26,

47)(27, 36)(31, 53)(32, 42)(33, 41)(34, 46)

(35, 51)(40, 54)(49, 52)

$= t_1$

$t_1^{(t_3t_0t_2t_1t_3t_2t_1t_0)}$ = (1, 9)(2, 55)(3, 16)(4, 6)(5, 17)(7, 31)(8,

32)(10, 22)(14, 26)(15, 39)(18, 33)(19, 49)

$$= (20, 48)(21, 40)(23, 45)(24, 46)(25, 41)(27, 50)(28, 30)(29, 51)(34, 54)(35, 43)(36, 44)(37, 42)(38, 52)(47, 53)$$

$$= t_2$$

$$t_2^{(t_3t_0t_2t_1t_3t_2t_1t_0)} = (1, 8)(2, 11)(3, 14)(4, 55)(9, 21)(10, 34)(12, 36)(13, 25)(15, 27)(16, 40)(17, 41)(18, 42)(19, 43)(20, 35)(22, 47)(23, 38)(24, 48)(26, 32)(28, 49)(29, 45)(30, 46)(31, 44)(33, 51)(37, 52)(39, 53)(50, 54)$$

$$= t_0$$

$$t_3^{(t_3t_0t_2t_1t_3t_2t_1t_0)} = (1, 10)(2, 13)(3, 55)(4, 7)(5, 29)(6, 18)(8, 20)(9, 33)(11, 35)(12, 24)(17, 47)(19, 37)(21, 50)(22, 31)(23, 49)(25, 34)(26, 46)(27, 42)(28, 44)(30, 52)(32, 53)(36, 54)(38, 41)(39, 43)(40, 45)(48, 51)$$

$$= t_3.$$

Thus $t_3t_0t_2t_1t_3t_2t_1t_0$ acts as the permutation (0 1 2) on the symmetric generators, that is

$$t_3t_0t_2t_1t_3t_2t_1t_0 = (0\ 1\ 2)$$

$$\Rightarrow t_0t_1t_2t_3 = (0\ 2\ 1)t_3t_0t_2t_1.$$

Therefore, our third relation holds true in $L_2(11)$.

The group $L_2(11)$ is generated by x, y, and t_0, and the relations hold true in $L_2(11)$. As a result, $L_2(11)$ is an image of G, therefore

$$|G| \geq |L_2(11)| = 660.$$

But from previous we know that $|G| \leq 660$, and so

$$|G| \leq 660 = |L_2(11)| \leq |G|,$$

which proves the isomorphism. Hence, $G \cong L_2(11)$.

CHAPTER THREE

SYMMETRIC REPRESENTAION OF $PGL_2(11)$ ELEMENTS

Introduction

In this chapter we will show that the group $PGL_2(11)$ is generated by 6 involutions which are permuted, by conjugation within $PGL_2(11)$, by a subgroup isomorphic to $L_2(5)$, and that each element of $PGL_2(11)$ can be written (not necessarily uniquely) as a permutation of $L_2(5)$ on 6 letters followed by a word of length less than or equal to three in these generators.

Factoring the progenitor $2^{*6} : L_2(5)$ by the relation

$$[(0\ 1\ 2\ 3\ 4)t_0]^4$$

yields the finite homomorphic image:

$$G \cong \frac{2^{*6} : L_2(5)}{[(0\ 1\ 2\ 3\ 4)t_0]^4},$$

where the index of $L_2(5)$ in G equals 22. $G \cong PGL_2(11)$, the projective general linear group.

A symmetric presentation for the above image is given by:

<x,y,t|x^5, y^3, (x*y)^2, t^2, (t, (y*x)^(x^2*y^2)),

(t, (x^3)^((y*x)^(y^2*x^3*y))), (x*t)^4>,

where $N = L_2(5) \cong$ <x,y|x^5, y^3, (x*y)^2>,

and the action of x, y on the symmetric generators is given by

$$x \sim (0\ 1\ 2\ 3\ 4)$$

$$y \sim (\infty\ 0\ 1)(2\ 4\ 3).$$

Double Coset Enumeration

The double coset [0] contains 6 distinct single cosets, namely Nt_0, Nt_1, Nt_2, Nt_3, Nt_4, Nt_∞.

Now, expanding the relation $[(0\ 1\ 2\ 3\ 4)t_0]^4 = 1$ yields:

$(\pi t_0)^4 = 1$, where $\pi = (0\ 1\ 2\ 3\ 4) \in N$

$\Rightarrow \pi^4(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}(t_0) = 1$

$\Rightarrow (0\ 4\ 3\ 2\ 1)t_3t_2t_1t_0 = 1$

$\Rightarrow \boxed{t_0t_1 = (0\ 4\ 3\ 2\ 1)t_3t_2 \quad (1)}$

$\Rightarrow Nt_0t_1 = Nt_3t_2$

Conjugating (1) by $(0\ \infty)(1\ 4) \in N$

$\Rightarrow t_\infty t_4 = (\infty\ 1\ 3\ 2\ 4)t_3t_2$

$\Rightarrow Nt_\infty t_4 = Nt_3t_2$

Therefore, $Nt_0t_1 = Nt_3t_2 = Nt_\infty t_4$.

The double coset [0 1] has $6 \cdot 5 = 30$ single cosets. But since Nt_0t_1 has three names, then the number of distinct single cosets in [0 1] $= \frac{30}{3} = 10$.

Now, applying t_2 to $Nt_0t_1 = Nt_3t_2$ yields:

$Nt_0t_1t_2 = Nt_3t_2t_2$

$= Nt_3$

but $t_3 \in [0]$ (since $t_3^{\pi} = t_0$, where $\pi = (0\ 3)(1\ 4) \in N$).

Therefore, $[0\ 1\ 2] = [0]$.

However, since $N^{(0\ 1)}$ has orbits $\{1, 2, 4\}$ and $\{0, 3, \infty\}$,

then $[0\ 1\ i\ ] = [0]$ for all i in $\{1, 2, 4\}$.

Now, $Nt_\infty t_3 t_\infty$ can be rewritten as

$Nt_\infty t_3 t_\infty = Nt_\infty \underline{t_3 t_\infty t_1 t_4} t_4 t_1$

$= Nt_\infty(4\ 1\ \infty\ 3\ 0)t_4t_1$ (since $t_3t_\infty t_1t_4 = (4\ 1\ \infty\ 3\ 0)$)

$= N\underline{t_3t_4}t_1$

$= Nt_1t_0t_1$ (since $Nt_3t_4 = Nt_1t_0$),

and also as

$Nt_\infty t_3 t_\infty = Nt_\infty \underline{t_3 t_\infty t_0 t_2} t_2 t_0$

$= Nt_\infty(2\ 0\ \infty\ 3\ 1)t_2t_0$ (since $t_3t_\infty t_0t_2 = (2\ 0\ \infty\ 3\ 1)$)

$= N\underline{t_3t_2}t_0$

$= Nt_0t_1t_0$ (since $Nt_3t_2 = Nt_0t_1$)

$\Rightarrow Nt_0t_1t_0 = Nt_1t_0t_1$.

$\Rightarrow Nt_it_jt_i = Nt_jt_it_j$ for i, j in $\{0, 1, 2, 3, 4, \infty\}$.

Therefore, $Nt_\infty t_3 t_\infty = Nt_3t_\infty t_3$.

Furthermore $Nt_2t_4t_2$ can be rewritten as

$Nt_2t_4t_2 = Nt_2\underline{t_4t_2t_\infty t_0}t_0t_\infty$

$= Nt_2(0\ \infty\ 2\ 4\ 3)t_0t_\infty$ (since $t_4t_2t_\infty t_0 = (0\ \infty\ 2\ 4\ 3)$)

$= Nt_4t_0t_\infty$

$= Nt_\infty t_3 t_\infty$ (since $Nt_4t_0 = Nt_\infty t_3$).

Therefore, $Nt_\infty t_3 t_\infty = Nt_3 t_\infty t_3 = Nt_0t_1t_0 = Nt_1t_0t_1 = Nt_2t_4t_2$

$= Nt_4t_2t_4$.

Therefore, the double coset [0 1 0] contains $6 \cdot 5 = 30$ single cosets, but since $Nt_0t_1t_0$ has 6 names, then the number of distinct single cosets in [0 1 0] $= \frac{30}{6} = 5$.

Furthermore, $N^{(0\ 1\ 0)}$ is transitive, which means that it has orbit {0, 1, 2, 3, 4, ∞}. As a result [0 1 0 j] = [0 1] for all j in {0, 1, 2, 3, 4, ∞}.

The set of all double cosets [w] = NwN, coset stabilizing subgroups $N^{(w)}$, and the number of single cosets are exhibited in Table 3.1. The table shows that the Cayley Graph of G over N has the form as displayed in Figure 3.1.

Table 3.1 The Double Cosets [w] = NwN in G, Where $N = L_2(5)$

NwN	Reason	Number of single cosets
[*]	N is transitive	1
[0]	$N^{(0)} = \langle (14)(23), (1324\infty) \rangle$ has orbits {0}, {1, 2, 3, 4, ∞}	6
[00]=[*]		

[01]	$N^{(0\ 1)}$ = <(24)(3∞), (12)(3∞), (124)(3∞0)> has orbits {1, 2, 4}, {0, 3, ∞}, 01 ~ 32 ~ 64 and for all i in {1, 2, 4}	10
[01i]=[0]		
[010]	$N^{(0\ 1\ 0)}$ = <(24)(3∞), (1∞2)(340), (1∞4)(203)> has orbit {0, 1, 2, 3, 4, ∞} and for all j in {0, 1, 2, 3, 4, ∞}	5
[010j]=[01]		

The Cayley graph of $PGL_2(11)$ over $L_2(5)$ has as vertices the set of right cosets of N in G, i.e. the set Nw_i, where w_i are words in T. The diagram below illustrates each orbit of N in its action on the vertices by right multiplication which is represented by a single node, labeled with the number of vertices that it contains. Lines in between these nodes are labeled with integers to denote how many edges from a vertex of one node will lead to the vertices of the other.

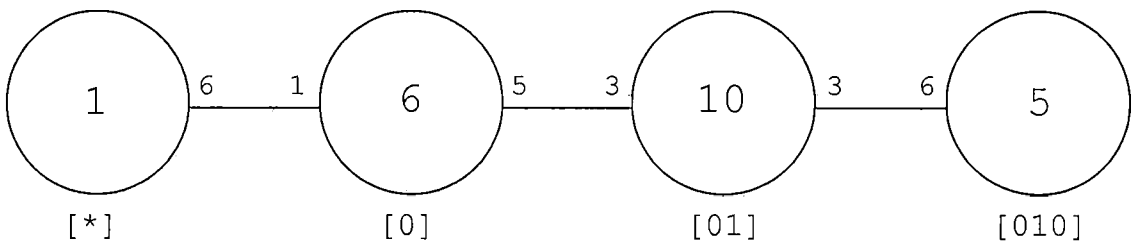


Figure 3.1 Cayley Graph of $PGL_2(11)$ over $L_2(5)$

At this point we can calculate the action of the six symmetric generators on the cosets of $PGL_2(11)$ over $L_2(5)$.

The labeling used for the single cosets of N in G is given by:

Label	Single Coset
22.	[N]
1.	[1]
2.	[2]
3.	[3]
4.	[4]
5.	[0]
6.	[∞]
7.	[0 1]
8.	[0 2]
9.	[0 3]
10.	[0 4]
11.	[0 ∞]
12.	[1 0]
13.	[1 2]
14.	[1 4]
15.	[2 0]
16.	[2 1]
17.	[0 1 3]
18.	[0 1 ∞]
19.	[0 2 3]
20.	[0 3 1]
21.	[0 3 2].

The action of the symmetric generators on the cosets of N in G is given by:

t_0 = (22,5)(1,12)(2,15)(3,14)(4,16)(6,13)(7,21)(8,20)(9,18)(10,19)(11,17)

t_1 = (22,1)(5,7)(2,16)(3,8)(4,15)(6,10)(9,20)(11,19)(12,21)(13,18)(14,17)

t_2 = (22,2)(5,8)(1,13)(3,7)(4,11)(6,12)(9,21)(10,17)(14,19)(15,20)(16,18)

t_3 = (22,3)(5,9)(1,11)(2,10)(4,13)(6,16)(7,17)(8,19)(12,20)(14,18)(15,21)

t_4 = (22,4)(5,10)(1,14)(2,9)(3,12)(6,7)(8,18)(11,21)(13,20)
(15,17)(16,19)

t_∞ = (22,6)(5,11)(1,9)(2,14)(3,15)(4,8)(7,18)(10,20)(12,19)
(13,17)(16,21).

Representation of Elements of $PGL_2(11)$

Every element of $PGL_2(11)$ can be represented as πw, where π is a permutation of $L_2(5)$, on 6 letters, and w is a product of at most three symmetric generators.

For example:

Let α = (1, 9, 19, 7, 3, 12, 17, 22, 20, 14)(2, 13, 5,
15, 6, 8, 18, 16, 21, 11)(4, 10),

for $\alpha \in PGL_2(11)$, where α is a permutation on 22 letters.

Now $N^\alpha = N\alpha$

$\Rightarrow$ $22^\alpha = 20$ (Since N is labeled as 22 and 22 goes to 20 under α)

$\Rightarrow$ $N\alpha = Nt_0t_3t_1$ (Since $Nt_0t_3t_1$ is labeled as 20)

$\Rightarrow$ $N\alpha t_1t_3t_0 = N$

$\Rightarrow$ $\alpha t_1t_3t_0 = n$, for $n \in L_2(5)$

$\Rightarrow$ $\alpha = nt_0t_3t_1$, a symmetric representation for α.

Now, what is n?

$\alpha t_1t_3t_0$ = (1, 9, 19, 7, 3, 12, 17, 22, 20, 14)(2, 13, 5, 15,
6, 8, 18, 16, 21, 11)(4, 10)(1, 9)(2, 14)(3, 5)
(4, 8)(5, 11)(6, 22)(7, 18)(10, 20)(12, 19)(13,

17)(16, 21)(1, 12)(2, 15)(3, 14)(4, 16)(5, 22)(6, 13)(7, 21)(8, 20)(9, 18)(10, 19)(11, 17)(1, 11)(2, 10)(3, 22)(4, 13)(5, 9)(6, 16)(7, 17)(8, 19)(12, 20)(14, 18)(15, 21)

$= (2, 3)(5, 6)(7, 10)(8, 16)(9, 12)(11, 13)(18, 19)(20, 21)$

Therefore, the action of $\alpha t_1 t_3 t_0$ on the symmetric generators is given by:

$Nt_1 = 1 \rightarrow 1 = Nt_1$

$Nt_2 = 2 \rightarrow 3 = Nt_3$

$Nt_3 = 3 \rightarrow 2 = Nt_2$

$Nt_4 = 4 \rightarrow 4 = Nt_4$

$Nt_0 = 5 \rightarrow 6 = Nt_\infty$

$Nt_\infty = 6 \rightarrow 5 = Nt_0$

$\Rightarrow n = (2\ 3)(0\ \infty)$.

Therefore, a symmetric representation for α is given by $\alpha = (2\ 3)(0\ \infty)t_0 t_3 t_1$, where $(2\ 3)(0\ \infty) \in L_2(5)$.

Permutation Representation of an Element Whose Symmetric Representation Is Given

Given a symmetric representation for any element of $PGL_2(11)$, we can easily recover its permutation representation on 22 lettters.

For example:

Let β = (1 ∞ 4 2 3)$t_\infty t_2 t_0$ be the symmetric representation of the element $\beta \in G$. Then the action of π = (1 ∞ 4 2 3) ∈ N on each of the twenty two single cosets of N = $L_2(5)$ in $PGL_2(11)$ is given by:

Label	Coset	action under π	Coset	Label
22 =	[N]	------>	[N] =	22
1 =	[1]	------>	[∞] =	6
2 =	[2]	------>	[3] =	3
3 =	[3]	------>	[1] =	1
4 =	[4]	------>	[2] =	2
5 =	[0]	------>	[0] =	5
6 =	[∞]	------>	[4] =	4
7 =	[0 1]	------>	[0 ∞] =	11
8 =	[0 2]	------>	[0 3] =	9
9 =	[0 3]	------>	[0 1] =	7
10 =	[0 4]	------>	[0 2] =	8
11 =	[0 ∞]	------>	[0 4] =	10
12 =	[1 0]	------>	[∞ 0] =	13
13 =	[1 2]	------>	[∞ 3] =	16
14 =	[1 4]	------>	[∞ 2] =	12
15 =	[2 0]	------>	[3 0] =	14
16 =	[2 1]	------>	[3 ∞] =	15
17 =	[0 1 3]	------>	[0 ∞ 1] =	19
18 =	[0 1 ∞]	------>	[0 ∞ 4] =	21
19 =	[0 2 3]	------>	[0 3 1] =	20
20 =	[0 3 1]	------>	[0 1 ∞] =	18
21 =	[0 3 2]	------>	[0 1 3] =	17

⇒ n = (1, 6, 4, 2, 3)(7, 11, 10, 8, 9)(12, 13, 16, 15, 14) (17, 19, 20, 18, 21).

So, a permutation representation for β is given by

β = (1, 6, 4, 2, 3)(7, 11, 10, 8, 9)(12, 13, 16, 15, 14) (17, 19, 20, 18, 21)$t_\infty t_2 t_0$

$= (1, 6, 4, 2, 3)(7, 11, 10, 8, 9)(12, 13, 16, 15, 14)$
$(17, 19, 20, 18, 21)(1, 9)(2, 14)(3, 15)(4, 8)(5, 11)$
$(6, 22)(7, 18)(10, 20)(12, 19)(13, 17)(16, 21)(1, 13)$
$(2, 22)(3, 7)(4, 11)(5, 8)(6, 12)(9, 21)(10, 17)(14,$
$19)(15, 20)(16, 18)(1, 12)(2, 15)(3, 14)(4, 16)(5,$
$22)(6, 13)(7, 21)(8, 20)(9, 18)(10, 19)(11, 17)$

$\Rightarrow \beta = (1, 15, 5, 16, 21, 12, 19, 11, 2, 8, 6, 22)(3, 7,$
$20, 14)(4, 10, 17, 13, 18, 9).$

Proof of Isomorphism

Double coset enumeration as displayed in Figure 3.1 shows that the group defined by the symmetric presentation contains a homomorphic image of $N \cong L_2(5)$ to index at most

$$\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(0\ 1)}|} + \frac{|N|}{|N^{(0\ 1\ 0)}|}$$

$$= \frac{60}{60} + \frac{60}{10} + \frac{60}{6} + \frac{60}{12}$$

$$= 1 + 6 + 10 + 5$$

$$= 22.$$

As a result, the index of N in G is at most 22.

Therefore, the order of the image group G is at most

$$22 \cdot |N| = 22 \cdot 60 = 1320.$$

Moreover, the action by right multiplication on the 22 cosets is implicit in the enumeration, and we can easily

verify that the image is the projective general linear group $PGL_2(11)$.

Furthermore, the order of G can be confirmed by regarding G as a permutation group on the 22 cosets that we have found. The action of x and y on the single cosets of N in G is given by:

x: (1, 2, 3, 4, 5)(7, 13, 10, 12, 16)(8, 11, 9, 14, 15)(17, 20, 19, 21, 18)

y: (1, 6, 5)(2, 4, 3)(7, 9, 13)(8, 14, 16)(10, 11, 12)(17, 21, 20).

As a result, the action of x and y on $\{t_0, t_1, t_2, t_3, t_4, t_\infty\}$ is given by:

x: $(t_0, t_1, t_2, t_3, t_4)$

y: $(t_\infty, t_0, t_1)(t_2, t_4, t_3)$.

The action of the symmetric generator t_0 on the single cosets of N in G is given by:

t_0: (22, 5)(1, 12)(2, 15)(3, 14)(4, 16)(6, 13)(7, 21)(8, 20)(9, 18)(10, 19)(11, 17).

Now, we need to verify that our relation holds true in $PGL_2(11)$.

Looking at our relation we can see that

$t_0^{(t_0t_1t_2t_3)}$ = (1, 14)(2, 9)(3, 12)(4, 22)(5, 10)(6, 7)(8, 18)

$$= (11, 21)(13, 20)(15, 17)(16, 19)$$

$$= t_4$$

$$t_1^{(t_0t_1t_2t_3)} = (1, 12)(2, 15)(3, 14)(4, 16)(5, 22)(6, 13)(7, 21)$$

$$(8, 20)(9, 18)(10, 19)(11, 17)$$

$$= t_0$$

$$t_2^{(t_0t_1t_2t_3)} = (1, 22)(2, 16)(3, 8)(4, 15)(5, 7)(6, 10)(9, 20)$$

$$(11, 19)(12, 21)(13, 18)(14, 17)$$

$$= t_1$$

$$t_3^{(t_0t_1t_2t_3)} = (1, 13)(2, 22)(3, 7)(4, 11)(5, 8)(6, 12)(9, 21)$$

$$(10, 17)(14, 19)(15, 20)(16, 18)$$

$$= t_2$$

$$t_4^{(t_0t_1t_2t_3)} = (1, 11)(2, 10)(3, 22)(4, 13)(5, 9)(6, 16)(7, 17)$$

$$(8, 19)(12, 20)(14, 18)(15, 21)$$

$$= t_3$$

$$t_\infty^{(t_0t_1t_2t_3)} = (1, 9)(2, 14)(3, 15)(4, 8)(5, 11)(6, 22)(7, 18)$$

$$(10, 20)(12, 19)(13, 17)(16, 21)$$

$$= t_\infty.$$

Thus, $t_0t_1t_2t_3$ acts as the permutation (0 4 3 2 1) on the symmetric generators, that is

$$t_0t_1t_2t_3 = (0\ 4\ 3\ 2\ 1)$$

$$\Rightarrow t_0t_1 = (0\ 4\ 3\ 2\ 1)t_3t_2.$$

Therefore, our relation holds true in $PGL_2(11)$.

The group $PGL_2(11)$ is generated by x, y and t_0, and the relation holds true in $PGL_2(11)$. As a result, $PGL_2(11)$ is an image of G, therefore

$$|G| \geq |PGL_2(11)| = 1320.$$

But from previous we know that $|G| \leq 1320$, and so

$$|G| \leq 1320 = |PGL_2(11)| \leq |G|,$$

which proves the isomorphism. Hence, $G \cong PGL_2(11)$.

CHAPTER FOUR

SYMMETRIC REPRESENTAION OF $PGL_2(7)$ ELEMENTS

Introduction

In this chapter we will show that the group $PGL_2(7)$ is generated by 4 involutions which are permuted, by conjugation within $PGL_2(7)$, by a subgroup isomorphic to A_4, and that each element of $PGL_2(7)$ can be written (not necessarily uniquely) as a permutation of A_4 on 4 letters followed by a word of length less than or equal to four in these generators.

Factoring the progenitor $2^{*4} : A_4$ by the following relations

$$[(0\ 1\ 2)t_0]^8,\ [(0\ 1)(2\ 3)t_0t_2]^3,\ \text{and}\ [(0\ 1)(2\ 3)t_0]^8$$

yields the finite homomorphic image:

$$G \cong \frac{2^{*4} : A_4}{[(0\ 1\ 2)t_0]^8,\ [(0\ 1)(2\ 3)t_0t_2]^3,\ [(0\ 1)(2\ 3)t_0]^8},$$

where the index of A_4 in G equals 28. $G \cong PGL_2(7)$, the projective general linear group.

A symmetric presentation for the above image is given by:

<x,y,t|x^3, y^3, (x*y)^2, t^2, (t,x), (y*t)^8,

(x*y*t*t^(y^2))^3>,

where $N = A_4 \cong$ <x,y|x^3, y^3, (x*y)^2>,

and the action of x, y on the symmetric generators is given by

$$x \sim (1\ 2\ 3)$$

$$y \sim (1\ 2\ 0).$$

Double Coset Enumeration

The double coset [0] has 4 distinct single cosets, namely Nt_0, Nt_1, Nt_2, and Nt_3. Meanwhile, the double coset [0 1] has 12 distinct single cosets, where $N^{(0\ 1)}$ has orbits {0}, {1}, {2}, and {3}.

Now, expanding the relation $[(0\ 1)(2\ 3))t_0t_2]^3 = 1$ yields:

$(\pi t_0t_2)^3 = 1$, where $\pi = (0\ 1)(2\ 3) \in N$

$\Rightarrow \pi^3(t_0t_2)^{\pi^2}(t_0t_2)^{\pi}(t_0t_2) = 1$

$\Rightarrow (0\ 1)(2\ 3)t_0t_2t_1t_3t_0t_2 = 1$

$\Rightarrow \boxed{t_2t_0t_3 = (0\ 1)(2\ 3)t_0t_2t_1 \quad (1)}$

$\Rightarrow t_0t_3t_1 = (0\ 1)(2\ 3)t_3t_0t_2$

$\Rightarrow [t_0t_3t_1]^{(1\ 2\ 3)} = [(0\ 1)(2\ 3)t_3t_0t_2]^{(1\ 2\ 3)}$

$\Rightarrow t_0t_1t_2 = (0\ 2)(3\ 1)t_1t_0t_3$

$\Rightarrow \boxed{Nt_0t_1t_2 = Nt_1t_0t_3. \quad (2)}$

The double coset [0 1 2] contains $4 \cdot 3 = 12$ single cosets. But since $Nt_0t_1t_2$ has two names, then the number of distinct single cosets in [0 1 2] $= \frac{12}{2} = 6.$

Applying t_3 to (2) yields:

$Nt_0t_1t_2t_3 = Nt_1t_0t_3t_3$

$= Nt_1t_0,$

but $Nt_1t_0 \in [0\ 1]$ (since $(Nt_1t_0)^{\pi} = Nt_0t_1,$

where $\pi = (0\ 1)(2\ 3) \in N$).

Therefore, $[0\ 1\ 2\ 3] = [0\ 1]$.

However, since $N^{(0\ 1\ 2)}$ has orbits $\{0, 1\}$ and $\{2, 3\}$ then

$[0\ 1\ 2\ i] = [0\ 1]$ for all i in $\{2, 3\}$.

Now, $t_0t_1t_2t_1$ can be written as

$t_0t_1t_2t_1 = t_0(3\ 1\ 0)t_3t_2t_3$

$= (3\ 1\ 0)t_3t_3t_2t_3$

$= (3\ 1\ 0)t_2t_3,$

but $t_2t_3 \in [0\ 1]$ (since $(t_2t_3)^{\pi} = t_0t_1,$

where $\pi = (0\ 2)(1\ 3)\ N$).

Therefore, $[0\ 1\ 2\ 1] = [0\ 1]$, which implies that $[0\ 1\ 2\ j]$

$= [0\ 1]$ for all j in $\{0, 1\}$.

Similarly, $[0\ 1\ 2] = [0\ 1\ 3]$.

Now, $Nt_0t_1t_0$ may be written as

$Nt_0t_1t_0 = Nt_0\underline{t_1t_0t_3t_2}t_2t_3$

$= Nt_0(0\ 2)(1\ 3)t_0t_1t_2t_3$

$= \underline{Nt_2t_0t_1t_2}t_3$

$= Nt_3t_1t_3,$

and $Nt_2t_1t_2$ may be written as

$Nt_2t_1t_2 = Nt_2\underline{t_1t_2t_3t_0}t_0t_3$

$= Nt_2(0\ 2)(1\ 3)t_2t_1t_0t_3$

$= \underline{Nt_0t_2t_1t_0}t_3$

$= Nt_3t_1t_3.$

Therefore, $Nt_0t_1t_0 = Nt_3t_1t_3 = Nt_2t_1t_2$.

The double coset [0 1 0] contains $4 \cdot 3 = 12$ single cosets.

But since $Nt_0t_1t_0$ has three names, then the number of

distinct single cosets in [0 1 0] $= \frac{12}{3} = 4$, namely, $Nt_0t_1t_0$,

$Nt_0t_2t_0$, $Nt_0t_3t_0$, and $Nt_3t_0t_3$, each with three names.

Now, applying t_3 to $Nt_0t_1t_0 = Nt_3t_1t_3$ yields:

$Nt_0t_1t_0t_3 = Nt_3t_1t_3t_3$

$= Nt_3t_1,$

but $Nt_3t_1 \in [0\ 1]$ (since $(Nt_3t_1)^{\pi} = Nt_0t_1$,

where $\pi = (0\ 2\ 3) \in N$).

Therefore, [0 1 0 3] = [0 1].

However, since $N^{(0\ 1\ 0)}$ has orbits {1} and {0, 2, 3}, then

[0 1 0 k] = [0 1] for all k in {0, 2, 3}.

Now, expanding the relation $[(0\ 1)(2\ 3)t_0]^8 = 1$ yields:

$(\pi t_0)^8 = 1$, where $\pi = (0\ 1)(2\ 3) \in N$

$\Rightarrow \pi^8(t_0)^{\pi^7}(t_0)^{\pi^6}(t_0)^{\pi^5}(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}(t_0) = 1$

$\Rightarrow t_1t_0t_1t_0t_1t_0t_1t_0 = 1$

$\Rightarrow \boxed{t_0t_1t_0t_1 = t_1t_0t_1t_0.} \qquad (3)$

So, $Nt_0t_1t_0t_1 = Nt_1t_0t_1t_0$

$= Nt_1(4\ 3\ 2)t_2t_1t_2$

$= Nt_1t_2t_1t_2$

$= Nt_1(4\ 3\ 2)t_3t_1t_3$

$= Nt_1t_3t_1t_3,$

and $Nt_0t_1t_0t_1 = Nt_1t_0t_1t_0$

$= Nt_3t_0t_3t_0$

$= Nt_2t_0t_2t_0$

$= Nt_2(0\ 1\ 3)t_3t_2t_3$

$= Nt_2t_3t_2t_3.$

Furthermore,

$Nt_1t_0t_1t_0 = Nt_0t_1t_0t_1$

$= Nt_0(1\ 2\ 3)t_1t_0t_1$

$= Nt_0t_3t_0t_3$

$= Nt_0(1\ 2\ 3)t_2t_0t_2$

$= Nt_0t_2t_0t_2,$

and $Nt_1t_0t_1t_0 = Nt_0t_1t_0t_1$

$= Nt_2t_1t_2t_1$

$= Nt_3t_1t_3t_1$

$= Nt_3(0\ 2\ 1)t_2t_3t_2$

$= Nt_3t_2t_3t_2.$

So, $Nt_0t_1t_0t_1 = Nt_2t_1t_2t_1 = Nt_3t_1t_3t_1 = Nt_0t_2t_0t_2 = Nt_1t_2t_1t_2 =$

$Nt_3t_2t_3t_2 = Nt_0t_3t_0t_3 = Nt_1t_3t_1t_3 = Nt_2t_3t_2t_3 = Nt_1t_0t_1t_0 =$

$Nt_2t_0t_2t_0 = Nt_3t_0t_3t_0$.

The double coset [0 1 0 1] contains $4 \cdot 3 = 12$ single cosets. But since $Nt_0t_1t_0t_1$ has twelve names as described above, then the number of distinct single cosets in the double coset [0 1 0 1] = 12/12 = 1.

Furthermore, applying t_0 to (3) yields:

$t_0t_1t_0t_1t_0 = t_1t_0t_1t_0t_0$

$= t_1t_0t_1$,

but $t_1t_0t_1 \in$ [0 1 0] (since $(Nt_1t_0t_1)^{\pi} = Nt_0t_1t_0$, where $\pi = (0\ 1)(2\ 3) \in N$).

Therefore, [0 1 0 1 0] = [0 1 0].

However, since $N^{(0\ 1\ 0\ 1)}$ has orbit {0, 1, 2, 3}, then [0 1 0 1 m] = [0 1 0] for all m in {0, 1, 2, 3}.

The set of all double cosets [w] = NwN, coset stabilizing subgroups $N^{(w)}$, and the number of single cosets are exhibited in Table 4.1. The table shows that the Cayley Graph of G over N has the form as displayed in Figure 4.1.

Table 4.1 The Double Cosets [w] = NwN in G, Where $N = A_4$

NwN	Reason	Number of single cosets
[*]	N is transitive	1
[0]	$N^{(0)} = \langle(123)\rangle$ has orbits {0}, {1, 2, 3}	4

[00]=[*]		
[01]	$N^{(0\ 1)}$ = <*e*> has orbits {0}, {1}, {2}, {3}	12
[012]	$N^{(0\ 1\ 2)}$ = <(01)(23)> has orbits {0, 1}, {2, 3} 012 = (02)(13)103 and for all i in {2, 3}	6
[012i]=[01]		
	012 = (031)231 and $(231)^{\pi}$ = 013 where π = (02)(13) $\in$ N	
[012]=[013]		
[010]	$N^{(0\ 1\ 0)}$ = <(230)> has orbits {1}, {0, 2, 3} 010 ~ 313 ~ 212 and for all j in {0, 2, 3}	4
[010j]=[01]		
[0101]	$N^{(0\ 1\ 0\ 1)}$ = <(230), (123)> has orbit {0, 1, 2, 3} and for all k in {0, 1, 2, 3}	1
[0101k]=[010]		

The Cayley graph of $PGL_2(7)$ over A_4 has as vertices the set of right cosets of N in G, i.e. the set Nw_i, where w_i are words in T. The diagram below illustrates each orbit of N in its action on the vertices by right multiplication which is represented by a single node, labeled with the number of vertices that it contains. Lines in between these nodes are labeled with integers to denote how many edges from a vertex of one node will lead to the vertices of the other. At this point we can calculate the action of the four symmetric generators on the cosets of $PGL_2(7)$ over A_4.

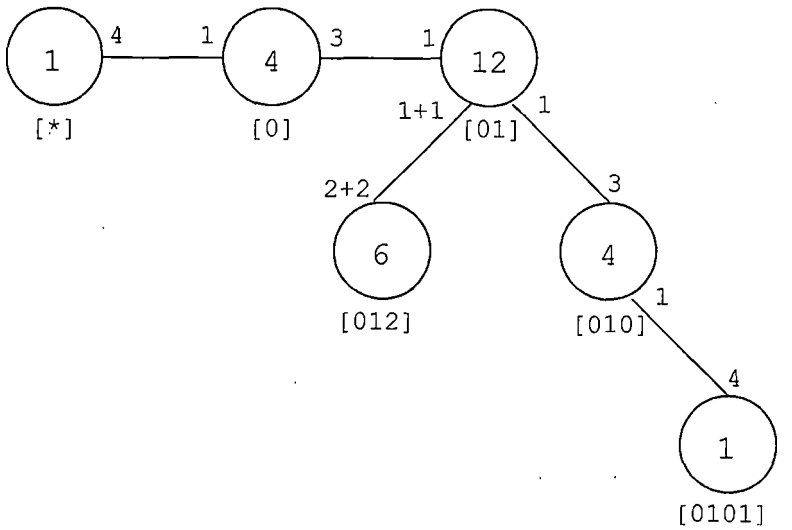


Figure 4.1 Cayley Graph of $PGL_2(7)$ over A_4

The labeling used for the single cosets of N in G is given by:

Label	Single Coset
28.	[N]
1.	[1]
2.	[2]
3.	[3]
4.	[0]
5.	[0 1]
6.	[0 2]
7.	[0 3]
8.	[1 0]
9.	[1 2]
10.	[1 3]
11.	[2 0]
12.	[2 1]
13.	[2 3]
14.	[3 0]
15.	[3 1]
16.	[3 2]
17.	[0 1 0]
18.	[0 2 0]
19.	[0 3 0]

Now $N^{\alpha} = N\alpha$

$\Rightarrow$ $28^{\alpha} = 23$ (Since N is labeled as 28 and 28 goes to 23 under α)

$\Rightarrow$ $N\alpha = Nt_1t_3t_2$ (Since $Nt_1t_3t_2$ is labeled as 23)

$\Rightarrow$ $N\alpha t_2t_3t_1 = N$

$\Rightarrow$ $\alpha t_2t_3t_1 = n$, for $n \in A_4$

$\Rightarrow$ $\alpha = nt_1t_3t_2$, a symmetric representation for α.

Now, what is n?

$\alpha t_2t_3t_1$ = (1, 12, 26, 10, 17, 9, 21, 15)(2, 7, 22, 28, 23, 6, 3, 8)(4, 16, 20, 13)(5, 18, 14, 25, 27, 24, 11, 19)(1, 9)(2, 28)(3, 16)(4, 6)(5, 21)(7, 25)(8, 26)(10, 23)(11, 20)(12, 17)(13, 19)(14, 22)(15, 24)(18, 27)(1, 10)(2, 13)(3, 28)(4, 7)(5, 26)(6, 24)(8, 21)(9, 22)(11, 23)(12, 25)(14, 20)(15, 17)(16, 18)(19, 27)(1, 28)(2, 12)(3, 15)(4, 5)(6, 23)(7, 22)(8, 20)(9, 18)(10, 19)(11, 24)(13, 21)(14, 25)(16,26)(17, 27)

= (1, 3, 4)(5, 10, 14)(6, 9, 16)(7, 8, 15)(11, 12, 13)(17, 19, 20)(21, 23, 22)(24, 25, 26)

Therefore, the action of $\alpha t_2t_3t_1$ on the symmetric generators is given by:

$Nt_1 = 1 \rightarrow 3 = Nt_3$

$Nt_2 = 2 \rightarrow 2 = Nt_2$

$Nt_3 = 3 \rightarrow 4 = Nt_0$

$Nt_0 = 4 \rightarrow 1 = Nt_1$

$\Rightarrow n = (1\ 3\ 0)$.

Therefore, a symmetric representation for α is given by $\alpha = (1\ 3\ 0)t_1t_3t_2$, where $(1\ 3\ 0) \in A_4$.

Permutation Representation of an Element Whose Symmetric Representation Is Given

Given a symmetric representation for any element of $PGL_2(7)$, we can easily recover its permutation representation on 28 lettters.

For example:

Let $\beta = (1\ 2\ 3)t_2t_0t_1$ be the symmetric representation of the element $\beta \in G$. Then the action of $\pi = (1\ 2\ 3) \in N$ on each of the twenty eight single cosets of $N = A_4$ in $PGL_2(7)$ is given by:

Label	Single Coset	action under π	Coset	Single Label
28.	[N]	------>	[N]	= 28
1.	[1]	------>	[2]	= 2
2.	[2]	------>	[3]	= 3
3.	[3]	------>	[1]	= 1
4.	[0]	------>	[0]	= 4
5.	[0 1]	------>	[0 2]	= 6
6.	[0 2]	------>	[0 3]	= 7
7.	[0 3]	------>	[0 1]	= 5
8.	[1 0]	------>	[2 0]	= 11
9.	[1 2]	------>	[2 3]	= 13
10.	[1 3]	------>	[2 1]	= 12
11.	[2 0]	------>	[3 0]	= 14

12. [2 1] ------> [3 2] = 16
13. [2 3] ------> [3 1] = 15
14. [3 0] ------> [1 0] = 8
15. [3 1] ------> [1 2] = 9
16. [3 2] ------> [1 3] = 10
17. [0 1 0] ------> [0 2 0] = 18
18. [0 2 0] ------> [0 3 0] = 19
19. [0 3 0] ------> [0 1 0] = 17
20. [3 0 3] ------> [1 0 1] = 20
21. [0 1 2] ------> [0 2 3] = 24
22. [0 3 1] ------> [0 1 2] = 21
23. [1 3 2] ------> [2 1 3] = 25
24. [2 0 1] ------> [3 0 2] = 22
25. [2 1 3] ------> [3 2 1] = 26
26. [3 2 1] ------> [1 3 2] = 23
27. [0 1 0 1] -----> [0 2 0 2] = 27

$\Rightarrow$ n = (1, 2, 3)(5, 6, 7)(8, 11, 14)(9, 13, 15)(10, 12, 16)

(17, 18, 19)(21, 24, 22)(23, 25, 26).

So, a permutation representation for β is given by

β = (1, 2, 3)(5, 6, 7)(8, 11, 14)(9, 13, 15)(10, 12, 16)

(17, 18, 19)(21, 24, 22)(23, 25, 26)$t_2t_0t_1$

= (1, 2, 3)(5, 6, 7)(8, 11, 14)(9, 13, 15)(10, 12, 16)

(17, 18, 19)(21, 24, 22)(23, 25, 26) (1, 9)(2, 28)(3,

16)(4, 6)(5, 21)(7, 25)(8, 26)(10, 23)(11, 20)(12, 17)

(13, 19)(14, 22)(15, 24)(18, 27) (1, 8)(2, 11)(3, 14)

(4, 28)(5, 17)(6, 18)(7, 19)(9, 25)(10, 24)(12, 22)

(13, 26)(15, 23)(16, 21)(20, 27) (1, 28)(2, 12)(3, 15)

(4, 5)(6, 23)(7, 22)(8, 20)(9, 18)(10, 19)(11, 24)(13,

21)(14, 25)(16, 26)(17, 27)

The action of x and y on the single cosets of N in G is given by:

x: (1, 2, 3)(5, 6, 7)(8, 11, 14)(9, 13, 15)(10, 12, 16)(17, 18,19)

y: (1, 2, 4)(5, 9, 11)(6, 8, 12)(7, 10, 13)(14, 15, 16)(17, 18, 20)(21, 25, 24)(22, 23, 26).

As a result, the action of x and y on $\{t_0, t_1, t_2, t_3\}$ is given by:

x: (t_1, t_2, t_3)

y: (t_1, t_2, t_0).

The action of the symmetric generator t_0 on the single cosets of N in G is given by:

t_0: (28, 4)(1, 8)(2, 11)(3, 14)(5, 17)(6, 18)(7, 19)(9, 25) (10, 24)(12, 22)(13, 26)(15, 23)(16, 21)(20, 27).

We note that the order of the product xy equals 2, and so $N = \langle x, y \rangle \cong A_4$.

Now, we need to verify that our three relations hold true in $PGL_2(7)$.

Looking at our first relation we can see that

$t_0^{(t_1t_0t_2t_1t_0t_2t_1t_0)}$ = (1, 28)(2, 12)(3, 15)(4, 5)(6, 23)(7, 22)(8, 20)(9, 18)(10, 19)(11, 24)(13, 21)(14, 25)(16, 26)(17, 27)

$= t_1$

$t_1^{(t_1t_0t_2t_1t_0t_2t_1t_0)}$ = (1, 9)(2, 28)(3, 16)(4, 6)(5, 21)(7, 25)(8, 26)(10, 23)(11, 20)(12, 17)(13, 19)(14, 22)(15, 24)(18, 27)

$= t_2$

$t_2^{(t_1t_0t_2t_1t_0t_2t_1t_0)}$ = (1, 8)(2, 11)(3, 14)(4, 28)(5, 17)(6, 18)(7, 19)(9, 25)(10, 24)(12, 22)(13, 26)(15, 23)(16, 21)(20, 27)

$= t_0$

$t_3^{(t_1t_0t_2t_1t_0t_2t_1t_0)}$ = (1, 10)(2, 13)(3, 28)(4, 7)(5, 26)(6, 24)(8, 21)(9, 22)(11, 23)(12, 25)(14, 20)(15, 17)(16, 18)(19, 27)

$= t_3$.

Thus, $t_1t_0t_2t_1t_0t_2t_1t_0$ acts as the permutation (1 2 0) on the symmetric generators, that is

$$t_1t_0t_2t_1t_0t_2t_1t_0 = (1\ 2\ 0)$$

$$\Rightarrow t_0t_1t_2t_0 = (1\ 0\ 2)t_1t_0t_2t_1.$$

Therefore, our first relation holds true in $PGL_2(7)$.

Now, looking at our second relation we can see that

$t_0^{(t_0t_2t_1t_3t_0t_2)}$ = (1, 28)(2, 12)(3, 15)(4, 5)(6, 23)(7, 22)(8, 20)(9, 18)(10, 19)(11, 24)(13, 21)(14, 25)(16, 26)(17, 27)

$= t_1$

$t_1^{(t_0t_2t_1t_3t_0t_2)}$ = (1, 8)(2, 11)(3, 14)(4, 28)(5, 17)(6, 18)(7, 19)(9, 25)(10, 24)(12, 22)(13, 26)(15, 23)(16, 21)(20, 27)

$= t_0$

$t_2^{(t_0t_2t_1t_3t_0t_2)}$ = (1, 10)(2, 13)(3, 28)(4, 7)(5, 26)(6, 24)(8, 21)(9, 22)(11, 23)(12, 25)(14, 20)(15, 17)(16, 18)(19, 27)

$= t_3$

$t_3^{(t_0t_2t_1t_3t_0t_2)}$ = (1, 9)(2, 28)(3, 16)(4, 6)(5, 21)(7, 25)(8, 26)(10, 23)(11, 20)(12, 17)(13, 19)(14, 22)(15, 24)(18, 27)

$= t_2$.

Thus, $t_0t_2t_1t_3t_0t_2$ acts as the permutation (0 1)(2 3) on the symmetric generators, that is

$$t_0t_2t_1t_3t_0t_2 = (0\ 1)(2\ 3)$$

$$\Rightarrow t_2t_0t_3 = (0\ 1)(2\ 3)t_0t_2t_1.$$

Therefore, our second relation holds true in $PGL_2(7)$.

Now, looking at our third relation we can see that

$t_0^{(t_1t_0t_1t_0t_1t_0t_1t_0)}$ = (1, 8)(2, 11)(3, 14)(4, 28)(5, 17)(6, 18)(7, 19)(9, 25)(10, 24)(12, 22)(13, 26)(15, 23)(16, 21)(20, 27)

$= t_0$

$t_1^{(t_1t_0t_1t_0t_1t_0t_1t_0)} = (1, 28)(2, 12)(3, 15)(4, 5)(6, 23)(7, 22)(8, 20)(9, 18)(10, 19)(11, 24)(13, 21)(14, 25)(16, 26)(17, 27)$

$= t_1$

$t_2^{(t_1t_0t_1t_0t_1t_0t_1t_0)} = (1, 9)(2, 28)(3, 16)(4, 6)(5, 21)(7, 25)(8, 26)(10, 23)(11, 20)(12, 17)(13, 19)(14, 22)(15, 24)(18, 27)$

$= t_2$

$t_3^{(t_1t_0t_1t_0t_1t_0t_1t_0)} = (1, 10)(2, 13)(3, 28)(4, 7)(5, 26)(6, 24)(8, 21)(9, 22)(11, 23)(12, 25)(14, 20)(15, 17)(16, 18)(19, 27)$

$= t_3.$

Thus, $t_1t_0t_1t_0t_1t_0t_1t_0$ acts as the identity on the symmetric generators, that is

$$t_1t_0t_1t_0t_1t_0t_1t_0 = Id$$

$$\Rightarrow t_0t_1t_0t_1 = t_1t_0t_1t_0.$$

Therefore, our third relation holds true in $PGL_2(7)$.

The group $PGL_2(7)$ is generated by x, y and t_0, and the relations hold true in $PGL_2(7)$. As a result, $PGL_2(7)$ is an image of G, therefore

$$|G| \geq |PGL_2(7)| = 336.$$

But from previous we know that $|G| \leq 336$, and so

$$|G| \leq 336 = |PGL_2(7)| \leq |G|,$$

which proves the isomorphism. Hence, $G \cong PGL_2(7)$.

CHAPTER FIVE

SYMMETRIC REPRESENTAION OF $U_3(3)$: 2 ELEMENTS

Introduction

In this chapter we will show that the group $U_3(3)$: 2 is generated by 7 involutions which are permuted, by conjugation within $U_3(3)$: 2, by a subgroup isomorphic to $L_3(2)$, and that each element of $U_3(3)$: 2 can be written (not necessarily uniquely) as a permutation of $L_3(2)$ on 7 letters followed by a word of length less than or equal to five in these generators.

Factoring the progenitor 2^{*7} : $L_3(2)$ by the following relation

$$(t_1t_3)^2 = (2\ 6)(4\ 5)$$

yields the finite homomorphic image:

$$G \cong \frac{2^{*7} : L_3(2)}{(t_1t_3)^2 = (2\ 6)(4\ 5)},$$

where the index of $L_3(2)$ in G equals 72. $G \cong U_3(3)$: 2, the full group of automorphisms of the derived Chevalley Group $G_2(2)$.

A symmetric presentation for the above image is given by:

<x,y,t|x^7, y^2, (x*y)^3, (x,y)^4, t^2, (x*y,t^(x^4)),
(y,t^(x^3)), y = (t^x*t^(x^3))^2>,

where $N = L_3(2) \cong \langle x,y,t \mid x^7,\ y^2,\ (x*y)^3,\ (x,y)^4 \rangle$,

and the action of x, y on the symmetric generators is given by

$$x \sim (0\ 1\ 2\ 3\ 4\ 5\ 6)$$

$$y \sim (2\ 6)(4\ 5).$$

Double Coset Enumeration

The double coset [0] contains 7 distinct single cosets, namely Nt_0, Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, and Nt_6.

Now, expanding the relation $(t_1t_3)^2 = (2\ 6)(4\ 5)$ yields:

$t_1t_3t_1t_3 = (2\ 6)(4\ 5)$

$$\Rightarrow \boxed{t_1t_3 = (2\ 6)(4\ 5)t_3t_1 \qquad (1)}$$

$\Rightarrow Nt_1t_3 = Nt_3t_1$

$\Rightarrow Nt_0t_1 = Nt_1t_0$.

The double coset [0 1] contains $7 \cdot 6 = 42$ single cosets, but since Nt_0t_1 has 2 names, then the number of distinct single cosets in [0 1] $= \frac{42}{2} = 21$.

Applying t_0 to $Nt_0t_1 = Nt_1t_0$ yields

$Nt_0t_1t_0 = Nt_1t_0t_0$

$= Nt_1$,

but $Nt_1 \in [0]$ (since $(Nt_1)^\pi = Nt_0$,

where $\pi = (0\ 1)(2\ 5) \in N$).

Therefore, [0 1 0] = [0].

However, since $N^{(0\ 1)}$ has orbits {0, 1}, {2, 4, 5, 6}, and {3}, then [0 1 i] = [0] for all i in {0, 1}.

Now, applying t_0 to (1) on the left hand side yields:

$t_0t_1t_3 = t_0(2\ 6)(4\ 5)t_3t_1$

$\Rightarrow Nt_0t_1t_3 = Nt_0t_3t_1$

$= Nt_3t_0t_1$

$= Nt_3(2\ 5)(4\ 6)t_1t_0$

$= Nt_3t_1t_0$

$= Nt_1t_3t_0$

$= Nt_1(2\ 4)(5\ 6)t_0t_3$

$= Nt_1t_0t_3$.

So, $Nt_0t_1t_3 = Nt_3t_1t_0 = Nt_3t_0t_1 = Nt_1t_3t_0 = Nt_1t_0t_3 = Nt_0t_3t_1$.

Therefore, the double coset [0 1 3] contains 7 distinct single cosets, each with 6 names.

Applying t_0 to $Nt_0t_1t_3 = Nt_3t_1t_0$ yields:

$Nt_0t_1t_3t_0 = Nt_3t_1t_0t_0$

$= Nt_3t_1$,

but $Nt_3t_1 \in$ [0 1] (since $(Nt_3t_1)^{\pi} = Nt_0t_1$, where $\pi = (0\ 3)(5\ 6) \in N$).

Therefore, [0 1 3 0] = [0 1].

However, since $N^{(0\ 1\ 3)}$ has orbits {0, 1, 3} and {2, 4, 5, 6}, then [0 1 3 j] = [0 1] for all j in {0, 1, 3}.

Now, $t_0t_1t_3t_2$ can be written as

$t_0t_1t_3t_2 = t_0t_1(1\ 6)(0\ 4)t_2t_3$

$= (1\ 6)(0\ 4)t_4t_6t_2t_3,$

but $t_4t_6t_2t_3 \in [0\ 1\ 2\ 3]$ (since $(t_4t_6t_2t_3)^{\pi} = t_0t_1t_2t_3,$

where $\pi = (1\ 6)(0\ 4) \in N$).

Therefore, $[0\ 1\ 3\ 2] = [0\ 1\ 2\ 3]$, which implies $[0\ 1\ 3\ k] = [0\ 1\ 2\ 3]$ for all k in {2, 4, 5, 6}.

Now, conjugating (1) by $(0\ 3\ 1)(2\ 4\ 5) \in N$ yields:

$t_0t_1 = (4\ 6)(5\ 2)t_1t_0$

$\Rightarrow t_0t_1t_2 = (4\ 6)(5\ 2)t_1t_0t_2$

$\Rightarrow Nt_0t_1t_2 = Nt_1t_0t_2$

$= Nt_1(3\ 4)(1\ 5)t_2t_0$

$= Nt_5t_2t_0$

$= Nt_2t_5t_0$

$= Nt_2(3\ 6)(1\ 2)t_0t_5$

$= Nt_1t_0t_5$

$= Nt_0t_1t_5$

$= Nt_0(3\ 4)(0\ 2)t_5t_1$

$= Nt_2t_5t_1$

$= Nt_5t_2t_1.$

So, $Nt_0t_1t_2 = Nt_0t_1t_5 = Nt_1t_0t_5 = Nt_1t_0t_2 = Nt_5t_2t_0 = Nt_2t_5t_0 = Nt_5t_2t_1 = Nt_2t_5t_1.$

Therefore, the double coset [0 1 2] contains 21 distinct single cosets, each with 8 names.

Applying t_0 to $Nt_0t_1t_2 = Nt_2t_5t_0$ yields:

$Nt_0t_1t_2t_0 = Nt_2t_5t_0t_0$

$= Nt_2t_5,$

but $Nt_2t_5 \in [0\ 1]$ (since $(Nt_2t_5)^{\pi} = Nt_0t_1,$

where $\pi = (0\ 2)(1\ 5) \in N$).

Therefore, [0 1 2 0] = [0 1].

However, since $N^{(0\ 1\ 2)}$ has orbits {0, 1, 2, 5}, {3}, and {4, 6}, then [0 1 2 m] = [0 1] for all m in {0, 1, 2, 5}.

Now, $Nt_0t_1t_2 = Nt_1t_0t_2$

$\Rightarrow Nt_0t_1t_2t_3 = Nt_1t_0t_2t_3$

$= Nt_1(3\ 4)(1\ 5)t_2t_0t_3$

$= Nt_5t_2t_0t_3$

$= Nt_2t_5t_0t_3$

$= Nt_2(3\ 6)(1\ 2)t_0t_5t_3$

$= Nt_1t_0t_5t_3$

$= Nt_0t_1t_5t_3$

$= Nt_0(3\ 4)(0\ 2)t_5t_1t_3$

$= Nt_2t_5t_1t_3,$

and $Nt_1t_0t_2t_3 = Nt_0t_1t_2t_3$

$= Nt_0(1\ 3)(2\ 6)t_2t_1t_6$

$= Nt_0t_2t_1t_6$

$= Nt_2t_0t_1t_6$

$= Nt_2(4\ 6)(2\ 5)t_1t_0t_6$

$= Nt_5t_1t_0t_6$

$= Nt_1t_5t_0t_6$

$= Nt_2t_0t_5t_6$

$= Nt_0t_2t_5t_6$

$= Nt_0(4\ 6)(0\ 1)t_5t_2t_6$

$= Nt_1t_5t_2t_6$

$= Nt_5t_1t_2t_6.$

Also, $Nt_0t_1t_2t_3 = Nt_1t_0t_2t_3$

$= Nt_1(0\ 3)(2\ 4)t_2t_0t_4$

$= Nt_1t_2t_0t_4$

$= Nt_2t_1t_0t_4$

$= Nt_2(4\ 6)(2\ 5)t_0t_1t_4$

$= Nt_5t_0t_1t_4$

$= Nt_0t_5t_1t_4$

$= Nt_0(3\ 4)(0\ 2)t_1t_5t_4$

$= Nt_2t_1t_5t_4$

$= Nt_1t_2t_5t_4$

$= Nt_1(4\ 6)(0\ 1)t_5t_2t_4$

$= Nt_0t_5t_2t_4$

$= Nt_5t_0t_2t_4.$

Therefore, the single coset $Nt_0t_1t_2t_3$ has 24 names as described above. And so the double coset [0 1 2 3] has 7 distinct cosets, namely $Nt_0t_1t_2t_3$, $Nt_1t_2t_3t_4$, $Nt_1t_6t_0t_5$, $Nt_2t_3t_4t_5$, $Nt_3t_4t_5t_6$, $Nt_4t_2t_6t_1$, and $Nt_6t_3t_5t_4$.

Applying t_4 to $Nt_0t_1t_2t_3 = Nt_5t_0t_2t_4$ yields:

$Nt_0t_1t_2t_3t_4 = Nt_5t_0t_2t_4t_4$

$= Nt_5t_0t_2,$

but $Nt_5t_0t_2 \in [0\ 1\ 2]$ (since $(Nt_5t_0t_2)^{\pi} = Nt_0t_1t_2,$ where $\pi = (0\ 1\ 5)(3\ 6\ 4) \in N$).

Therefore, [0 1 2 3 4] = [0 1 2].

However, since $N^{(0\ 1\ 2\ 3)}$ has orbits {0, 1, 2, 5} and {3, 4, 6}, then [0 1 2 3 p]= [0 1 2] for all p in {3, 4, 6}.

Now, $t_0t_1t_2t_3t_2$ can be written as

$t_0t_1t_2t_3t_2 = t_0t_1t_2(1\ 6)(0\ 4)t_2t_3$

$= (1\ 6)(0\ 4)t_4t_6t_2t_2t_3$

$= (1\ 6)(0\ 4)t_4t_6t_3,$

but $t_4t_6t_3 \in [0\ 1\ 3]$ (since $(t_4t_6t_3)^{\pi} = t_0t_1t_3,$ where $\pi = (1\ 6)(0\ 4) \in N$).

Therefore, [0 1 2 3 2] = [0 1 3], which implies that [0 1 2 3 q] = [0 1 3] for all q in {0, 1, 2, 5}.

Now, $Nt_0t_1t_2 = Nt_1t_0t_5$

$\Rightarrow Nt_0t_1t_2t_4 = Nt_1t_0t_5t_4$

$= Nt_5t_2t_1t_4$

$= Nt_5t_2(0\ 6)(3\ 5)t_4t_1$

$= Nt_3t_2t_4t_1$

$= Nt_4t_0t_3t_1$

$= Nt_4(2\ 4)(5\ 6)t_3t_0t_1$

$= Nt_2t_3t_0t_1$

$= Nt_0t_4t_2t_1$

$= Nt_0(0\ 5)(3\ 6)t_4t_1t_2$

$= Nt_5t_4t_1t_2$

$= Nt_5(0\ 6)(3\ 5)t_1t_4t_2$

$= Nt_3t_1t_4t_2$

$= Nt_1t_3t_5t_2$

$= Nt_4t_5t_3t_2$

$= Nt_4t_2t_3t_5$

$= Nt_4(0\ 4)(1\ 6)t_5t_2t_3$

$= Nt_0t_5t_2t_3$

$= Nt_0(0\ 1)(4\ 6)t_2t_5t_3$

$= Nt_1t_2t_5t_3$

$= Nt_1(0\ 1)(4\ 6)t_3t_2t_5$

$= Nt_0t_3t_2t_5$

$= Nt_3t_0t_4t_5$

$= Nt_2t_4t_0t_5$

$= Nt_2(1\ 2)(3\ 6)t_4t_5t_0$

$= Nt_1t_4t_5t_0$

$= Nt_1(1\ 3)(2\ 6)t_5t_4t_0$

$= Nt_3t_5t_4t_0$

$= Nt_3(1\ 6)(2\ 3)t_5t_0t_4$

$= Nt_2t_5t_0t_4$

$= Nt_5t_2t_1t_4,$

and $Nt_0t_1t_2t_4 = Nt_4t_0t_3t_1$

$= Nt_4(2\ 6)(4\ 5)t_3t_1t_0$

$= Nt_5t_3t_1t_0$

$= Nt_5(2\ 6)(4\ 5)t_1t_3t_0$

$= Nt_4t_1t_3t_0$

$= Nt_4(2\ 6)(4\ 5)t_0t_1t_3$

$= Nt_5t_0t_1t_3$

$= Nt_5(2\ 5)(4\ 6)t_1t_0t_3$

$= Nt_2t_1t_0t_3.$

Therefore, the single coset $Nt_0t_1t_2t_4$ has 24 names as described above. And so the double coset [0 1 2 4] has 7 distinct cosets, namely $Nt_0t_1t_2t_4$, $Nt_1t_2t_3t_5$, $Nt_2t_3t_4t_6$, $Nt_2t_4t_3t_6$, $Nt_2t_5t_1t_6$, $Nt_3t_4t_5t_0$, and $Nt_4t_5t_6t_1$.

Applying t_0 to $Nt_0t_1t_2t_4 = Nt_4t_1t_3t_0$ yields:

$Nt_0t_1t_2t_4t_0 = Nt_4t_1t_3t_0t_0$

$\Rightarrow Nt_0t_1t_2t_4t_0 = Nt_4t_1t_3,$

but $Nt_4t_1t_3 \in [0\ 1\ 2]$ (since $(Nt_4t_1t_3)^{\pi} = Nt_0t_1t_2$, where $\pi = (2\ 3)(0\ 4) \in N$).

Therefore, [0 1 2 4 0] = [0 1 2].

However, since $N^{(0\ 1\ 2\ 4)}$ has orbits {0, 1, 2, 3, 4, 5} and {6}, then [0 1 2 4 r] = [0 1 2] for all r in {0, 1, 2, 3, 4, 5}.

Now, applying t_6 to $Nt_0t_1t_2t_4 = Nt_4t_1t_3t_0$ yields:

$$Nt_0t_1t_2t_4t_6 = Nt_4t_1t_3t_0t_6$$

$$= N(0\ 6\ 5)(1\ 4\ 2)t_0t_1t_3t_4$$

$$\Rightarrow Nt_0t_1t_2t_4t_6 = Nt_2t_0t_1t_3t_4.$$

Using the above relation to calculate the order of the stabilizing group $N^{(0\ 1\ 2\ 4\ 6)}$ (see Appendix A) we have that the order of this stabilizing group equals 168. Therefore, the number of distinct single cosets in the double coset [0 1 2 4 6] turns out to be $\frac{|N|}{|N^{(0\ 1\ 2\ 4\ 6)}|} = \frac{168}{168} = 1$, which means that all the single cosets in the double coset [0 1 2 4 6] are all equal to each other. Furthermore, $N^{(0\ 1\ 2\ 4\ 6)}$ is transitive, which means that [0 1 2 4 6 s] = [0 1 2 4] for all s in {0, 1, 2, 3, 4, 5, 6}.

The set of all double cosets [w] = NwN, coset stabilizing subgroups $N^{(w)}$, and the number of single cosets are exhibited in Table 5.1. The table shows that the Cayley Graph of G over N has the form as displayed in Figure 5.1.

Table 5.1 The Double Cosets [w] = NwN in G, Where N = $L_3(2)$

NwN	Reason	Number of single cosets
[*]	N is transitive	1
[0]	$N^{(0)}$ = <(26)(45), (142)(356), (25)(46)> has orbits {0}, {1, 2, 3, 4, 5, 6}	7
[00]=[*]		
[01]	$N^{(0\ 1)}$ = <(26)(45), (25)(46), (01)(46), (01)(25), (01)(2456), (01)(2654)> has orbits {0, 1}, {3}, {2, 4}, 01 = (46)(52)10 and for all i in {0, 1}	21
[01i]=[0]		
[013]	$N^{(0\ 1\ 3)}$ = <(26)(45), (25)(46), (01)(46), (01)(25), (01)(2456), (01)(2654), (2546)(03), (24)(03), (03)(56), (2645)(03)> has orbits {0, 1, 3}, {2, 4, 5, 6}, 013 ~ 031 ~ 103 ~ 130 ~ 301 ~ 310 and for all j in {0, 1, 3}	7
[013j]=[01]		
	0132 = 01(16)(04)23 = (16)(04)4623 and $(4623)^{\pi}$ = 0123 where π = (16)(04) $\in$ N	
[0132]=[0123]		
[012]	$N^{(0\ 1\ 2)}$ = <(01)(46), (1502)(46)> has orbits {0, 1, 2, 5}, {4, 6}, 012 ~ 102 ~ 251 ~ 521 ~ 105 ~ 015 ~ 520 ~ 250 and for all k in {0, 1, 2, 5}	21
[012k]=[012]		
[0123]	$N^{(0\ 1\ 2\ 3)}$ = <(01)(46), (1502)(46), (1520)(36)> has orbits {0, 1, 2, 5}, {3, 4, 6} and for all m in {3, 4, 6}	7
[0123m]=[012]		
	01232 = 012(16)(04)23 = (16)(04)463 and $(463)^{\pi}$ = 013 where π = (16)(04) $\in$ N	
[01232]=[013]		

[0124]	$N^{(0\ 1\ 2\ 4)}$ = <(15)(02), (01)(25), (130)(254)> has orbits {6}, {0, 1, 2, 3, 4, 5} and for all p in {0, 1, 2, 3, 4, 5}	7
[0124p]=[012]		
[01246]	$N^{(0\ 1\ 2\ 4\ 6)}$ = <(15)(02), (01)(25), (130)(254), (1543)(06)> has orbit {0, 1, 2, 3, 4, 5, 6} and for all q in {0, 1, 2, 3, 4, 5, 6}	1
[01246q]=[012]		

The Cayley graph of $U_3(3)$: 2 over $L_3(2)$ has as vertices the set of right cosets of N in G, i.e. the set Nw_i, where w_i are words in T. The diagram below illustrates each orbit of N in its action on the vertices by right multiplication which is represented by a single node, labeled with the number of vertices that it contains. Lines in between

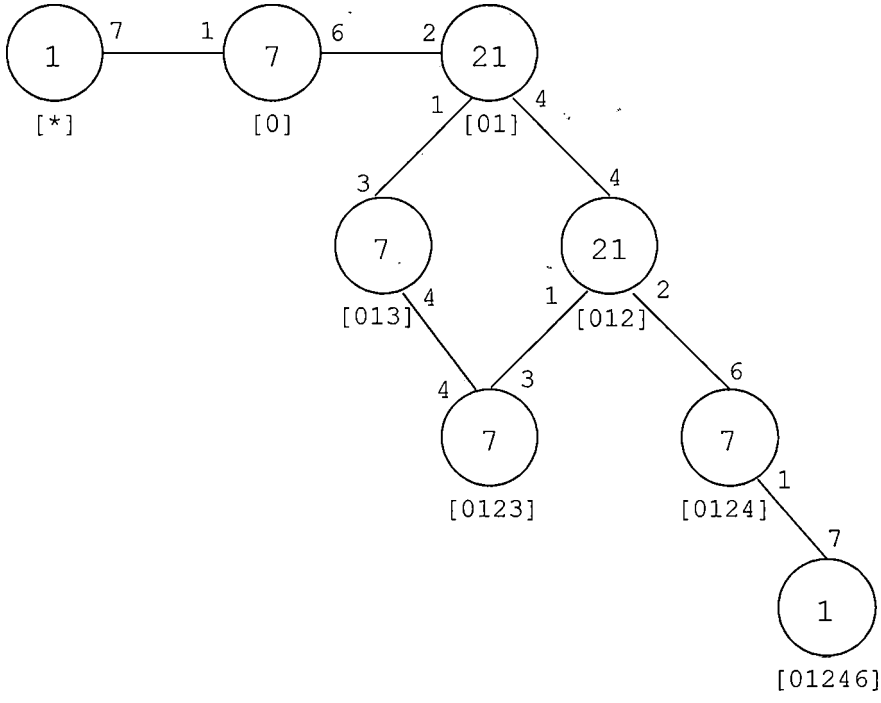


Figure 5.1 Cayley Graph of $U_3(3)$: 2 over $L_3(2)$

these nodes are labeled with integers to denote how many edges from a vertex of one node will lead to the vertices of the other.

At this point we can calculate the action of the seven symmetric generators on the cosets of $U_3(3) : 3$ over $L_3(2)$. The labeling used for the single cosets of N in G is given by:

Label	Single Coset
72.	[N]
1.	[1]
2.	[2]
3.	[3]
4.	[4]
5.	[5]
6.	[6]
7.	[0]
8.	[0 1]
9.	[0 2]
10.	[0 3]
11.	[0 4]
12.	[0 5]
13.	[0 6]
14.	[1 2]
15.	[1 3]
16.	[1 4]
17.	[1 5]
18.	[1 6]
19.	[2 3]
20.	[2 4]
21.	[2 5]
22.	[2 6]
23.	[3 4]
24.	[3 5]
25.	[3 6]
26.	[4 5]
27.	[4 6]
28.	[5 6]
29.	[0 1 2]
30.	[0 4 6]
31.	[0 5 2]
32.	[1 2 3]
33.	[1 4 0]
34.	[1 5 0]
35.	[1 6 3]
36.	[2 3 4]
37.	[2 6 1]
38.	[2 0 4]
39.	[3 1 5]
40.	[3 0 2]
41.	[3 0 6]
42.	[4 1 3]
43.	[5 1 3]
44.	[5 2 4]
45.	[5 3 0]
46.	[6 2 4]
47.	[6 3 5]
48.	[6 4 1]
49.	[6 5 4]
50.	[0 1 3]
51.	[1 2 4]
52.	[1 6 5]
53.	[2 3 5]
54.	[5 0 4]
55.	[6 0 2]
56.	[6 3 4]
57.	[0 1 2 3]

58. [1 2 3 4]	65. [1 2 3 5]
59. [1 6 0 5]	66. [2 3 4 6]
60. [2 3 4 5]	67. [2 4 3 6]
61. [3 4 5 6]	68. [2 5 1 6]
62. [4 2 6 1]	69. [3 4 5 0]
63. [6 3 5 4]	70. [4 5 6 1]
64. [0 1 2 4]	71. [0 1 2 4 6]

The action of the symmetric generators on the cosets of N in G is given by:

t_0 = (72,7)(1,8)(2,9)(3,10)(4,11)(5,12)(6,13)(14,31)(15,50)(16,33)(17,34)(18,30)(19,36)(20,40)(21,29)(22,55)(23,38)(24,45)(25,47)(26,54)(27,48)(28,41)(32,67)(35,68)(37,58)(39,61)(42,64)(43,69)(44,66)(46,62)(49,70)(51,63)(52,60)(53,59)(56,57)(65,71)

t_1 = (72,1)(7,8)(2,14)(3,15)(4,16)(5,17)(6,18)(9,34)(10,50)(11,30)(12,31)(13,33)(19,35)(20,51)(21,29)(22,37)(23,43)(24,42)(25,32)(26,39)(27,48)(28,52)(36,64)(38,67)(40,60)(41,63)(44,65)(45,68)(46,70)(47,69)(49,62)(53,59)(54,58)(55,61)(56,57)(66,71)

t_2 = (72,2)(7,9)(1,14)(3,19)(4,20)(5,21)(6,22)(8,29)(10,40)(11,36)(12,31)(13,55)(15,37)(16,51)(17,34)(18,35)(23,38)(24,53)(25,32)(26,46)(27,44)(28,49)(30,70)(33,59)(39,64)(41,68)(42,61)(43,65)(45,63)(47,66)(48,67)(50,62)(52,60)(54,58)(56,57)(69,71)

t_3 = (72,3)(7,10)(1,15)(2,19)(4,23)(5,24)(6,25)(8,50)(9,38)(11,36)(12,47)(13,45)(14,32)(16,42)(17,43)(18,35)(20,40)(21,53)(22,37)(26,39)(27,56)(28,41)(29,57)(30,69)(31,64)(33,67)(34,68)(44,62)(46,66)(48,59)(49,65)(51,63)(52,60)(54,58)(55,61)(70,71)

t_4 = (72,4)(7,11)(1,16)(2,20)(3,23)(5,26)(6,27)(8,48)(9,38)(10,40)(12,54)(13,33)(14,51)(15,39)(17,43)(18,30)(19,36)(21,44)(22,46)(24,42)(25,56)(28,49)(29,64)(31,57)(32,58)(34,70)(35,65)(37,67)(41,69)(45,66)(47,63)(50,62)(52,60)(53,59)(55,61)(68,71)

t_5 = (72,5)(7,12)(1,17)(2,21)(3,24)(4,26)(6,28)(8,29)(9,34)(10,41)(11,54)(13,45)(14,31)(15,39)(16,42)(18,52)(19,53)(20,49)(22,46)(23,43)(25,47)(27,44)(30,59)(32,65)

(33,70)(35,58)(36,60)(37,68)(38,66)(40,64)(48,69)(50, 62)(51,63)(55,61)(56,57)(67,71)

t_6 = (72,6)(7,13)(1,18)(2,22)(3,25)(4,27)(5,28)(8,48)(9,55) (10,41)(11,30)(12,47)(14,32)(15,37)(16,33)(17,52)(19, 35)(20,49)(21,44)(23,56)(24,45)(26,46)(29,68)(31,70) (34,57)(36,66)(38,60)(39,69)(40,67)(42,65)(43,61)(50, 62)(51,63)(53,59)(54,58)(64,71).

Representation of Elements of $U_3(3) : 2$

Every element of $U_3(3) : 2$ can be represented as πw, where π is a permutation of $L_3(2)$, on 7 letters, and w is a product of at most five symmetric generators.

For example:

Let α = (1, 62, 36, 22, 43, 11, 56, 68)(2, 60, 38, 27, 53, 8, 37, 58)(3, 13)(4, 66, 50, 18, 39, 9, 35, 57)(5, 12, 45, 28)(6, 24, 7, 25, 47, 72, 41, 10)(14, 71, 20, 49, 64, 33, 65, 31)(15, 48, 23, 55)(16, 40, 67, 32)(17, 29, 59, 44, 26, 54, 69, 52)(19, 30) (21, 34, 61, 46)(42, 70)(51, 63),

for $\alpha \in U_3(3) : 2$, where α is a permutation on 72 letters.

Now $N^{\alpha} = N\alpha$

$\Rightarrow$ $72^{\alpha} = 41$ (Since N is labeled as 72 and 72 goes to 41 under α)

$\Rightarrow$ $N\alpha = Nt_3t_0t_6$ (Since $Nt_3t_0t_6$ is labeled as 41)

$\Rightarrow$ $N\alpha t_6t_0t_3 = N$

$\Rightarrow$ $\alpha t_6t_0t_3 = n$, for $n \in L_3(2)$

$\Rightarrow \quad \alpha = nt_3t_0t_6$, a symmetric representation for α.

Now, what is n?

$\alpha t_6t_0t_3$ = (2, 4)(5, 6)(9, 11)(12, 13)(14, 16)(17, 18)(19, 23)(21, 27)(22, 26)(24, 25)(29, 48)(30, 34)(31, 33)(32, 42)(35, 43)(36, 38)(37, 39)(45, 47)(53, 56)(54, 55)(57, 59)(58, 61)(64, 67)(68, 69)

Therefore, the action of $\alpha t_6t_0t_3$ on the symmetric generators is given by:

$Nt_1 = 1 \rightarrow 1 = Nt_1$

$Nt_2 = 2 \rightarrow 4 = Nt_4$

$Nt_3 = 3 \rightarrow 3 = Nt_3$

$Nt_4 = 4 \rightarrow 2 = Nt_2$

$Nt_5 = 5 \rightarrow 6 = Nt_6$

$Nt_6 = 6 \rightarrow 5 = Nt_5$

$Nt_0 = 7 \rightarrow 7 = Nt_0$

$\Rightarrow$ n = (2 4)(5 6).

And so a symmetric representation for α is given by

α = (2 4)(5 6)$t_3t_0t_6$.

Permutation Representation of an Element Whose Symmetric Representation Is Given

Given a symmetric representation for any element of $U_3(3) : 2$, we can easily recover its permutation representation on 72 letters.

For example:

Let β = (2 6)(4 5)t_2t_1 be the symmetric representation of the element $\beta \in G$. Then the action of π = (2 6)(4 5) $\in$ N on each of the seventy two single cosets of N = $L_3(2)$ in $U_3(3)$: 2 is given by:

Label	Single Coset	action under π	Coset	Single Coset
72.	[N]	------>	[N] =	72
1.	[1]	------>	[1] =	1
2.	[2]	------>	[2] =	6
3.	[3]	------>	[3] =	3
4.	[4]	------>	[5] =	5
5.	[5]	------>	[4] =	4
6.	[6]	------>	[2] =	2
7.	[0]	------>	[0] =	7
8.	[0 1]	------>	[0 1] =	8
9.	[0 2]	------>	[0 6] =	13
10.	[0 3]	------>	[0 3] =	10
11.	[0 4]	------>	[0 5] =	12
12.	[0 5]	------>	[0 4] =	11
13.	[0 6]	------>	[0 2] =	9
14.	[1 2]	------>	[1 6] =	18
15.	[1 3]	------>	[1 3] =	15
16.	[1 4]	------>	[1 5] =	17
17.	[1 5]	------>	[1 4] =	16
18.	[1 6]	------>	[1 2] =	14
19.	[2 3]	------>	[6 3] =	25
20.	[2 4]	------>	[6 5] =	28
21.	[2 5]	------>	[6 4] =	27
22.	[2 6]	------>	[6 2] =	22
23.	[3 4]	------>	[3 5] =	24
24.	[3 5]	------>	[3 4] =	23
25.	[3 6]	------>	[3 2] =	19
26.	[4 5]	------>	[5 4] =	26
27.	[4 6]	------>	[5 2] =	21
28.	[5 6]	------>	[4 2] =	20
29.	[0 1 2]	------>	[0 1 6] =	48
30.	[0 4 6]	------>	[0 5 2] =	31
31.	[0 5 2]	------>	[0 4 6] =	30

32. [1 2 3] ------> [1 6 3] = 35
33. [1 4 0] ------> [1 5 0] = 34
34. [1 5 0] ------> [1 4 0] = 33
35. [1 6 3] ------> [1 2 3] = 32
36. [2 3 4] ------> [6 3 5] = 47
37. [2 6 1] ------> [6 2 1] = 37
38. [2 0 4] ------> [6 0 5] = 45
39. [3 1 5] ------> [3 1 4] = 39
40. [3 0 2] ------> [3 0 6] = 41
41. [3 0 6] ------> [3 0 2] = 40
42. [4 1 3] ------> [5 1 3] = 43
43. [5 1 3] ------> [4 1 3] = 42
44. [5 2 4] ------> [4 6 5] = 44
45. [5 3 0] ------> [4 3 0] = 38
46. [6 2 4] ------> [2 6 5] = 46
47. [6 3 5] ------> [2 3 4] = 36
48. [6 4 1] ------> [2 5 1] = 29
49. [6 5 4] ------> [2 4 5] = 49
50. [0 1 3] ------> [0 1 3] = 50
51. [1 2 4] ------> [1 6 5] = 52
52. [1 6 5] ------> [1 2 4] = 51
53. [2 3 5] ------> [6 3 4] = 56
54. [5 0 4] ------> [4 0 5] = 54
55. [6 0 2] ------> [2 0 6] = 55
56. [6 3 4] ------> [2 3 5] = 53
57. [0 1 2 3]------> [0 1 6 3] = 59
58. [1 2 3 4]------> [1 6 3 5] = 58
59. [1 6 0 5]------> [1 2 0 4] = 57
60. [2 3 4 5]------> [6 3 5 4] = 63
61. [3 4 5 6]------> [3 5 4 2] = 61
62. [4 2 6 1]------> [5 6 2 1] = 62
63. [6 3 5 4]------> [2 3 4 5] = 60
64. [0 1 2 4]------> [0 1 6 5] = 69
65. [1 2 3 5]------> [1 6 3 4] = 65
66. [2 3 4 6]------> [6 3 5 2] = 66
67. [2 4 3 6]------> [6 5 3 2] = 68
68. [2 5 1 6]------> [6 4 1 2] = 67
69. [3 4 5 0]------> [3 5 4 0] = 64
70. [4 5 6 1]------> [5 4 2 1] = 70
71. [0 1 2 4 6]----->[0 1 6 5 2] = 71

$\Rightarrow$ n = (2, 6)(4, 5)(9, 13)(11, 12)(14, 18)(16, 17)(19, 25)
(20, 28)(21, 27)(23, 24)(29, 48)(30, 31)(32, 35)(33,
34)(36, 47)(38, 45)(40, 41)(42, 43)(51, 52)(53, 56)
(57, 59)(60, 63)(64, 69)(67, 68).

So, a permutation representation for β is given by

β = (2, 6)(4, 5)(9, 13)(11, 12)(14, 18)(16, 17)(19, 25)
(20, 28)(21, 27)(23, 24)(29, 48)(30, 31)(32, 35)(33,
34)(36, 47)(38, 45)(40, 41)(42, 43)(51, 52)(53, 56)
(57, 59)(60, 63)(64, 69)(67, 68)t_2t_1

= (1, 2, 37, 3, 35, 32, 6)(4, 29, 38, 41, 50, 49, 52)(5,
51, 40, 45, 43, 55, 33)(7, 34, 53, 56, 42, 44, 48)(8,
21, 65, 23, 59, 57, 13)(9, 61, 24, 67, 63, 28, 16)(10,
60, 68, 27, 17, 20, 62)(11, 12, 64, 66, 69, 26, 70)(14,
19, 25, 15, 22, 18, 72)(30, 31, 46, 39, 36, 71, 47).

Proof of Isomorphism

Double coset enumeration as displayed in Figure 5.1 shows that the group defined by the symmetric presentation contains a homomorphic image of $N \cong L_3(2)$ to index at most

$$\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(0\ 1)}|} + \frac{|N|}{|N^{(0\ 1\ 2)}|} + \frac{|N|}{|N^{(0\ 1\ 3)}|} +$$

$$\frac{|N|}{|N^{(0\ 1\ 2\ 3)}|} + \frac{|N|}{|N^{(0\ 1\ 2\ 4)}|} + \frac{|N|}{|N^{(0\ 1\ 2\ 4\ 6)}|}$$

$$= \frac{168}{168} + \frac{168}{24} + \frac{168}{8} + \frac{168}{8} + \frac{168}{24} + \frac{168}{24} + \frac{168}{24} + \frac{168}{168}$$

= 1 + 7 + 21 + 21 + 7 + 7 + 7 + 1

= 72.

As a result, the index of N in G is at most 72.

Therefore, the order of the image group G is at most

$$72 \cdot |N| = 72 \cdot 168 = 12096.$$

Moreover, the action by right multiplication on the 72 cosets is implicit in the enumeration, and we can verify that the image is the $U_3(3) : 2$.

Furthermore, the order of G can by confirmed by regarding G as a permutation group on the 72 cosets that we have found. The action of x and y on the single cosets of N in G is given by:

x: (1, 2, 3, 4, 5, 6, 7)(8, 14, 19, 23, 26, 28, 13)(9, 15, 20, 24, 27, 12, 18)(10, 16, 21, 25, 11, 17, 22)(29, 32, 36, 43, 46, 41, 33)(30, 34, 37, 40, 42, 44, 47)(31, 35, 38, 39, 49, 45, 48)(50, 51, 53, 56, 54, 52, 55)(57, 58, 60, 61, 62, 63, 59)(64, 65, 66, 69, 70, 68, 67)

y: (2, 6)(4, 5)(9, 13)(11, 12)(14, 18)(16, 17)(19, 25)(20, 28)(21, 27)(23, 24)(29, 48)(30, 31)(32, 35)(33, 34)(36, 47)(38, 45)(40, 41)(42, 43)(51, 52)(53, 56)(57, 59)(60, 63)(64, 69)(67, 68).

The action of the symmetric generator t_0 on the single cosets of N in G is given by:

t_0: (72, 7)(1, 8)(2, 9)(3, 10)(4, 11)(5, 12)(6, 13)(14, 31)(15, 50)(16, 33)(17, 34)(18, 30)(19, 36)(20, 40)(21, 29)(22, 55)(23, 38)(24, 45)(25, 47)(26, 54)(27, 48)(28,

41)(32, 67)(35, 68)(37, 58)(39, 61)(42, 64)(43, 69)(44, 66)(46, 62)(49, 70)(51, 63)(52, 60)(53, 59)(56, 57)(65, 71).

Now, we need to verify that our relation holds true in $U_3(3) : 2$.

Looking at our relation we can see that

$t_0^{(t_1t_3)^2}$ = (1, 8)(2, 9)(3, 10)(4, 11)(5, 12)(6, 13)(7, 72)(14, 31)(15, 50)(16, 33)(17, 34)(18, 30)(19, 36)(20, 40)(21, 29)(22, 55)(23, 38)(24, 45)(25, 47)(26, 54)(27, 48)(28, 41)(32, 67)(35, 68)(37, 58)(39, 61)(42, 64)(43, 69)(44, 66)(46, 62)(49, 70)(51, 63)(52, 60)(53, 59)(56, 57)(65, 71)

= t_0

$t_1^{(t_1t_3)^2}$ = (1, 72)(2, 14)(3, 15)(4, 16)(5, 17)(6, 18)(7, 8)(9, 34)(10, 50)(11, 30)(12, 31)(13, 33)(19, 35)(20, 51)(21, 29)(22, 37)(23, 43)(24, 42)(25, 32)(26, 39)(27, 48)(28, 52)(36, 64)(38, 67)(40, 60)(41, 63)(44, 65)(45, 68)(46, 70)(47, 69)(49, 62)(53, 59)(54, 58)(55, 61)(56, 57)(66, 71)

= t_1

$t_2^{(t_1t_3)^2}$ = (1, 18)(2, 22)(3, 25)(4, 27)(5, 28)(6, 72)(7, 13)(8, 48)(9, 55)(10, 41)(11, 30)(12, 47)(14, 32)(15,

37)(16, 33)(17, 52)(19, 35)(20, 49)(21, 44)(23, 56)
(24, 45)(26, 46)(29, 68)(31, 70)(34, 57)(36, 66)
(38, 60)(39, 69)(40, 67)(42, 65)(43, 61)(50, 62)
(51, 63)(53, 59)(54, 58)(64, 71)

$= t_6$

$t_3^{(t_1t_3)^2}$ = (1, 15)(2, 19)(3, 72)(4, 23)(5, 24)(6, 25)(7, 10)
(8, 50)(9, 38)(11, 36)(12, 47)(13, 45)(14, 32)(16,
42)(17, 43)(18, 35)(20, 40)(21, 53)(22, 37)(26, 39)
(27, 56)(28, 41)(29, 57)(30, 69)(31, 64)(33, 67)
(34, 68)(44, 62)(46, 66)(48, 59)(49, 65)(51, 63)
(52, 60)(54, 58)(55, 61)(70, 71)

$= t_3$

$t_4^{(t_1t_3)^2}$ = (1, 17)(2, 21)(3, 24)(4, 26)(5, 72)(6, 28)(7, 12)
(8, 29)(9, 34)(10, 41)(11, 54)(13, 45)(14, 31)(15,
39)(16, 42)(18, 52)(19, 53)(20, 49)(22, 46)(23, 43)
(25, 47)(27, 44)(30, 59)(32, 65)(33, 70)(35, 58)
(36, 60)(37, 68)(38, 66)(40, 64)(48, 69)(50, 62)
(51, 63)(55, 61)(56, 57)(67, 71)

$= t_5$

$t_5^{(t_1t_3)^2}$ = (1, 16)(2, 20)(3, 23)(4, 72)(5, 26)(6, 27)(7, 11)
(8, 48)(9, 38)(10, 40)(12, 54)(13, 33)(14, 51)(15,
39)(17, 43)(18, 30)(19, 36)(21, 44)(22, 46)(24, 42)

$$(25, 56)(28, 49)(29, 64)(31, 57)(32, 58)(34, 70)$$
$$(35, 65)(37, 67)(41, 69)(45, 66)(47, 63)(50, 62)$$
$$(52, 60)(53, 59)(55, 61)(68, 71)$$
$$= t_4$$

$$t_6^{(t_1t_3)^2} = (1, 14)(2, 72)(3, 19)(4, 20)(5, 21)(6, 22)(7, 9)(8,$$
$$29)(10, 40)(11, 36)(12, 31)(13, 55)(15, 37)(16, 51)$$
$$(17, 34)(18, 35)(23, 38)(24, 53)(25, 32)(26, 46)$$
$$(27, 44)(28, 49)(30, 70)(33, 59)(39, 64)(41, 68)$$
$$(42, 61)(43, 65)(45, 63)(47, 66)(48, 67)(50, 62)$$
$$(52, 60)(54, 58)(56, 57)(69, 71)$$
$$= t_2.$$

Thus, $(t_1t_3)^2$ acts as the permutation (2 6)(4 5) on the symmetric generators, that is

$$t_1t_3 = (2\ 6)(4\ 5)t_3t_1.$$

Therefore, our relation holds true in $U_3(3) : 2$.

The group $U_3(3) : 2$ is generated by x, y, and t_0, and the relations hold true in $U_3(3) : 2$. As a result, $U_3(3) : 2$ is an image of G, therefore

$$|G| \geq |U_3(3) : 2| = 12096.$$

But from previous we know that $|G| \leq 12096$, and so

$$|G| \leq 12096 = |U_3(3) : 2| \leq |G|,$$

which proves the isomorphism. Hence, $G \cong U_3(3) : 2$.

CHAPTER SIX

SYMMETRIC REPRESENTAION OF $L_2(13)$ ELEMENTS

Introduction

In this chapter we will show that the group $L_2(13)$ is generated by 4 involutions which are permuted, by conjugation within $L_2(13)$, by a subgroup isomorphic to A_4, and that each element of $L_2(13)$ can be written (not necessarily uniquely) as a permutation of A_4 on 4 letters followed by a word of length less than or equal to five in these generators.

Factoring the progenitor $2^{*4} : A_4$ by the following relations

$$[(0\ 1\ 2)t_0]^7,\ [(0\ 1)(2\ 3)t_0]^7, \text{ and } [(0\ 1\ 2)t_0t_1t_2t_3]^2$$

yields the finite homomorphic image:

$$G \cong \frac{2^{*4} : A_4}{[(0\ 1\ 2)t_0]^7,\ [(0\ 1)(2\ 3)t_0]^7,\ [(0\ 1\ 2)t_0t_1t_2t_3]^2},$$

where A_4 is maximal in $L_2(13)$ and the index of A_4 in G equals 91. $G \cong L_2(13)$, the projective special linear group.

A symmetric presentation for the above image is given by:

<x,y,t|x^3, y^3, (x*y)^2, t^2, (t,x), (y*t)^7,

(x*y*t)^7,(y*t*t^y*t^(y^2)*t^(y^2*x))^2>,

where $N = A_4 \cong$ <x,y|x^3, y^3, (x*y)^2>,

and the action of x, y on the symmetric generators is given by

$$x \sim (1\ 2\ 3)$$

$$y \sim (1\ 2\ 0).$$

Double Coset Enumeration

The double coset [0] has 4 distinct single cosets, namely, Nt_0, Nt_1, Nt_2, and Nt_3. Meanwhile, the double cosets [0 1] and [0 1 0] each contain 12 distinct single cosets.

Now, expanding the relation $[(0\ 1)(2\ 3)t_0]^7 = 1$ yields:

$(\pi t_0)^7 = 1$, where $\pi = (0\ 1)(2\ 3) \in N$

$$\Rightarrow \pi^7 (t_0)^{\pi^6} (t_0)^{\pi^5} (t_0)^{\pi^4} (t_0)^{\pi^3} (t_0)^{\pi^2} (t_0)^{\pi} (t_0) = 1$$

$$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1t_0t_1t_0 = 1$$

$$\Rightarrow \boxed{t_0t_1t_0t_1 = (0\ 1)(2\ 3)t_0t_1t_0 \quad (1)}$$

$$\Rightarrow Nt_0t_1t_0t_1 = Nt_0t_1t_0$$

$$\Rightarrow [0\ 1\ 0\ 1] = [0\ 1\ 0].$$

The double coset [0 1 0 2] has 12 distinct single cosets.

Now, $Nt_0t_1t_0t_2t_1$ can be written as

$$Nt_0t_1t_0t_2t_1 = Nt_0(0\ 2\ 1)t_1t_2t_0$$

$$= Nt_2t_1t_2t_0,$$

but $Nt_2t_1t_2t_0 \in [0\ 1\ 0\ 3]$ (since $(Nt_2t_1t_2t_0)^{\pi} = Nt_0t_1t_0t_3$, where $\pi = (0\ 3\ 2) \in N$).

Therefore, [0 1 0 2 1] = [0 1 0 3].

Now, $t_0t_1t_0t_2t_3$ can be written as

$t_0t_1t_0t_2t_3 = t_0(0\ 2\ 3)t_2t_0t_3t_1$

$= (0\ 2\ 3)t_2t_2t_0t_3t_1$

$= (0\ 2\ 3)t_0t_3t_1,$

but $t_0t_3t_1 \in [0\ 1\ 2]$ (since $(t_0t_3t_1)^{\pi} = t_0t_1t_2$ where $\pi = (1\ 2\ 3) \in N$).

Therefore, [0 1 0 2 3] = [0 1 2].

Now, applying t_2t_0 to (1) yields:

$Nt_0t_1t_0t_2t_0 = Nt_0t_1t_0t_1t_2t_0$

$= Nt_0t_1(0\ 1\ 2)t_0t_2t_1$

$= Nt_1t_2t_0t_2t_1$

$= Nt_1t_0t_1t_3t_1.$

Therefore, $Nt_0t_1t_0t_2t_0 = Nt_1t_0t_1t_3t_1$.

Furthermore, the double coset [0 1 0 2 0] contains $4 \cdot 3 \cdot 1 \cdot 2 \cdot 1 = 12$ single cosets. But since $Nt_0t_1t_0t_2t_0$ has two names, then the number of distinct single cosets in [0 1 0 2 0] equals $\frac{12}{2} = 6$, namely $Nt_0t_1t_0t_2t_0$, $Nt_3t_2t_3t_1t_3$, $Nt_2t_0t_2t_1t_2$, $Nt_2t_1t_2t_3t_2$, $Nt_1t_3t_1t_2t_1$, and $Nt_0t_3t_0t_1t_0$, each with two names.

Applying t_1 to $Nt_0t_1t_0t_2t_0 = Nt_1t_0t_1t_3t_1$ yields

$Nt_0t_1t_0t_2t_0t_1 = Nt_1t_0t_1t_3t_1t_1$

$= Nt_1t_0t_1t_3,$

but $Nt_1t_0t_1t_3 \in [0\ 1\ 0\ 2]$ (since $(Nt_1t_0t_1t_3)^{\pi} = Nt_0t_1t_0t_2$, where $\pi = (0\ 1)(2\ 3) \in N$).

Therefore, $[0\ 1\ 0\ 2\ 0\ 1] = [0\ 1\ 0\ 2]$.

However, since $N^{(0\ 1\ 0\ 2)}$ has orbits $\{0, 1\}$ and $\{2, 3\}$ then $[0\ 1\ 0\ 2\ 0\ i] = [0\ 1\ 0\ 2]$ for all i in $\{0, 1\}$.

Furthermore, $[0\ 1\ 0\ 3\ 0] = [0\ 1\ 0\ 2\ 0]$.

Now, $t_0t_1t_0t_2t_0t_2$ can be written as

$t_0t_1t_0t_2t_0t_2 = t_0t_1(0\ 2)(1\ 3)t_0t_2t_0$

$= (0\ 2)(1\ 3)t_2t_3t_0t_2t_0$

$= (0\ 2)(1\ 3)t_2(0\ 1\ 3)t_3t_2t_3t_1$

$= (0\ 2\ 1)t_2t_3t_2t_3t_1$

$= (0\ 3\ 2)t_2t_3t_2t_1,$

but $t_2t_3t_2t_1 \in [0\ 1\ 0\ 3]$ (since $(t_2t_3t_2t_1)^{\pi} = t_0t_1t_0t_3$ where $\pi = (0\ 2)(1\ 3) \in N$).

Therefore, $[0\ 1\ 0\ 2\ 0\ 2] = [0\ 1\ 0\ 3]$ which implies that $[0\ 1\ 0\ 2\ 0\ j] = [0\ 1\ 0\ 3]$ for all j in $\{2, 3\}$.

The double coset $[0\ 1\ 0\ 3]$ has 12 distinct single cosets.

Now, $Nt_0t_1t_0t_3t_1$ can be written as

$Nt_0t_1t_0t_3t_1 = Nt_0(0\ 3\ 1)t_1t_3t_0$

$= Nt_3t_1t_3t_0,$

but $Nt_3t_1t_3t_0 \in [0\ 1\ 0\ 2]$ (since $(Nt_3t_1t_3t_0)^{\pi} = Nt_0t_1t_0t_2$, where $\pi = (0\ 2\ 3) \in N$).

Therefore, [0 1 0 3 1] = [0 1 0 2].

Now, $t_0t_1t_0t_3t_2$ can be written as

$t_0t_1t_0t_3t_2 = t_0(0\ 3\ 2)t_3t_0t_2t_1$

$= (0\ 3\ 2)t_3t_3t_0t_2t_1$

$= (0\ 3\ 2)t_0t_2t_1,$

but $t_0t_2t_1 \in [0\ 1\ 3]$ (since $(t_0t_2t_1)^\pi = t_0t_1t_3$

where $\pi = (2\ 1\ 3) \in N$).

Therefore, [0 1 0 3 2] = [0 1 3].

The double coset [0 1 2] has twelve distinct single cosets.

Expanding the relation $[(0\ 1\ 2)t_0]^7 = 1$ yields:

$(\pi t_0)^7 = 1$, where $\pi = (0\ 1\ 2) \in N$

$\Rightarrow \pi^7(t_0)^{\pi^6}(t_0)^{\pi^5}(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}(t_0) = 1$

$\Rightarrow (0\ 1\ 2)t_0t_2t_1t_0t_2t_1t_0 = 1$

$\Rightarrow \boxed{t_0t_1t_2t_0 = (0\ 1\ 2)t_0t_2t_1.} \quad (2)$

$\Rightarrow Nt_0t_1t_2t_0 = Nt_0t_2t_1,$

but $Nt_0t_2t_1 \in [0\ 1\ 3]$ (since $(Nt_0t_2t_1)^\pi = Nt_0t_1t_3,$

where $\pi = (1\ 3\ 2) \in N$).

Therefore, [0 1 2 0] = [0 1 3].

Applying t_2 to (1) yields:

$Nt_0t_1t_0t_2 = Nt_0t_1t_0t_1t_2$

$= Nt_0(1\ 2\ 3)t_1t_3t_0t_3$

$= Nt_0t_1t_3t_0t_3$

$= Nt_0t_3t_1t_3,$

but $Nt_0t_3t_1t_3 \in [0\ 1\ 2\ 1]$ (since $(Nt_0t_3t_1t_3)^{\pi} = Nt_0t_1t_2t_1$, where $\pi = (3\ 1\ 2) \in N$).

Therefore, $[0\ 1\ 2\ 1] = [0\ 1\ 0\ 2]$.

Now, expanding the relation $[(0\ 1\ 2)t_0t_1t_2t_3]^2 = 1$ yields:

$(\pi t_0t_1t_2t_3)^2 = 1$, where $\pi = (0\ 1\ 2) \in N$

$\Rightarrow \pi^2(t_0t_1t_2t_3)^{\pi}(t_0t_1t_2t_3) = 1$

$\Rightarrow (0\ 2\ 1)t_1t_2t_0t_3t_0t_1t_2t_3 = 1$

$\Rightarrow \boxed{t_0t_1t_2t_3 = (0\ 1\ 2)t_3t_1t_0t_2 \quad (3)}$

$\Rightarrow (Nt_0t_1t_2t_3)^{(0\ 3\ 2)} = (Nt_3t_1t_0t_2)^{(0\ 3\ 2)}$

$\Rightarrow Nt_3t_1t_0t_2 = Nt_2t_1t_3t_0.$

Therefore, $Nt_0t_1t_2t_3 = Nt_3t_1t_0t_2 = Nt_2t_1t_3t_0$.

The double coset $[0\ 1\ 2\ 3]$ contains $4 \cdot 3 \cdot 2 \cdot 1 = 12$ single cosets. But since $Nt_0t_1t_2t_3$ has three names, then the number of distinct single cosets in $[0\ 1\ 2\ 3] = \frac{12}{3} = 4$, namely $Nt_0t_1t_2t_3$, $Nt_0t_2t_3t_1$, $Nt_0t_3t_1t_2$, and $Nt_3t_0t_2t_1$, each with three names.

Now, $Nt_0t_1t_2t_3t_1$ can be written as

$Nt_0t_1t_2t_3t_1 = Nt_0t_1(1\ 2\ 3)t_2t_1t_3t_2$

$= Nt_0t_2t_2t_1t_3t_2$

$= Nt_0t_1t_3t_2.$

Therefore, $[0\ 1\ 2\ 3\ 1] = [0\ 1\ 3\ 2]$.

Applying t_0 to $Nt_0t_1t_2t_3 = Nt_2t_1t_3t_0$ yields:

$Nt_0t_1t_2t_3t_0 = Nt_2t_1t_3t_0t_0$

$= Nt_2t_1t_3,$

but $Nt_2t_1t_3 \in [0\ 1\ 2]$ (since $(Nt_2t_1t_3)^{\pi} = Nt_0t_1t_2,$

where $\pi = (0\ 3\ 2) \in N$).

Therefore, $[0\ 1\ 2\ 3\ 0] = [0\ 1\ 2]$.

However, since $N^{(0\ 1\ 2\ 3)}$ has orbits $\{1\}$ and $\{2, 3, 0\}$ then

$[0\ 1\ 2\ 3\ k] = [0\ 1\ 2]$ for all k in $\{2, 3, 0\}$.

The double coset $[0\ 1\ 3]$ has 12 distinct single cosets.

Now, conjugating (2) by $(1\ 3\ 2) \in N$ yields:

$[t_0t_1t_2t_0]^{(1\ 3\ 2)} = [(0\ 1\ 2)t_0t_2t_1]^{(1\ 3\ 2)}$

$\Rightarrow t_0t_3t_1t_0 = (0\ 3\ 1)t_0t_1t_3$

$\Rightarrow t_0t_3t_1 = (0\ 3\ 1)t_0t_1t_3t_0$

$\Rightarrow Nt_0t_3t_1 = Nt_0t_1t_3t_0,$

but $Nt_0t_3t_1 \in [0\ 1\ 2]$ (since $(Nt_0t_3t_1)^{\pi} = Nt_0t_1t_2,$

where $\pi = (1\ 2\ 3) \in N$).

Therefore, $[0\ 1\ 3\ 0] = [0\ 1\ 2]$.

Applying t_3 to (1) yields:

$Nt_0t_1t_0t_3 = Nt_0t_1t_0t_1t_3$

$= Nt_0(1\ 3\ 2)t_1t_2t_0t_2$

$= Nt_0t_1t_2t_0t_2$

$= Nt_0t_2t_1t_2,$

but $Nt_0t_2t_1t_2 \in [0\ 1\ 3\ 1]$ (since $(Nt_0t_2t_1t_2)^{\pi} = Nt_0t_1t_3t_1$, where $\pi = (1\ 3\ 2) \in N$).

Therefore, $[0\ 1\ 3\ 1] = [0\ 1\ 0\ 3]$.

From (3) we have

$t_0t_1t_2t_3 = (0\ 1\ 2)t_3t_1t_0t_2$

$\Rightarrow t_3(0\ 2\ 1)t_0t_1t_2 = t_1t_0t_2t_3$

$\Rightarrow Nt_3t_0t_1t_2 = Nt_1t_0t_2t_3$

$\Rightarrow (Nt_3t_0t_1t_2)^{(0\ 1)(2\ 3)} = (Nt_1t_0t_2t_3)^{(0\ 1)(2\ 3)}$

$\Rightarrow Nt_2t_1t_0t_3 = Nt_0t_1t_3t_2$

$\Rightarrow (Nt_2t_1t_0t_3)^{(0\ 3\ 2)} = (Nt_0t_1t_3t_2)^{(0\ 3\ 2)}$

$\Rightarrow Nt_0t_1t_3t_2 = Nt_3t_1t_2t_0$.

Therefore, $Nt_0t_1t_3t_2 = Nt_2t_1t_0t_3 = Nt_3t_1t_2t_0$.

The double coset $[0\ 1\ 3\ 2]$ contains $4 \cdot 3 \cdot 2 \cdot 1 = 12$ single cosets. But since $Nt_0t_1t_3t_2$ has three names, then the number of distinct single cosets in $[0\ 1\ 3\ 2] = \frac{12}{3} = 4$, namely $Nt_0t_1t_3t_2$, $Nt_0t_2t_1t_3$, $Nt_0t_3t_2t_1$, and $Nt_3t_0t_1t_2$, each with three names.

Now, $Nt_0t_1t_3t_2t_1$ can be written as

$Nt_0t_1t_3t_2t_1 = Nt_0t_1(1\ 3\ 2)t_3t_1t_2t_3$

$= Nt_0t_3t_3t_1t_2t_3$

$= Nt_0t_1t_2t_3$.

Therefore, $[0\ 1\ 3\ 2\ 1] = [0\ 1\ 2\ 3]$.

Applying t_0 to $Nt_0t_1t_3t_2 = Nt_3t_1t_2t_0$ yields:

$Nt_0t_1t_3t_2t_0 = Nt_3t_1t_2t_0t_0$

$= Nt_3t_1t_2,$

but $Nt_3t_1t_2 \in [0\ 1\ 3]$ (since $(Nt_3t_1t_2)^{\pi} = Nt_0t_1t_3$,

where $\pi = (0\ 2\ 3) \in N$).

Therefore, $[0\ 1\ 3\ 2\ 0] = [0\ 1\ 3]$.

However, since $N^{(0\ 1\ 3\ 2)}$ has orbits {1} and {2, 3, 0} then

$[0\ 1\ 3\ 2\ m] = [0\ 1\ 3]$ for all m in {2, 3, 0}.

The set of all double cosets [w] = NwN, coset stabilizing subgroups $N^{(w)}$, and the number of single cosets are exhibited in Table 6.1. The table shows that the Cayley Graph of G over N has the form as displayed in Figure 6.1.

Table 6.1 The Double Cosets [w] = NwN in G, Where N = A_4

NwN	Reason	Number of single cosets
[*]	N is transitive	1
[0]	$N^{(0)} = \langle(123)\rangle$ has orbits {0}, {1, 2, 3}	4
[00]=[*]		
[01]	$N^{(0\ 1)} = \langle e \rangle$ has orbits {0}, {1}, {2}, {3}	12
[010]	$N^{(0\ 1\ 0)} = \langle e \rangle$ has orbits {0}, {1}, {2}, {3}, and 0101 = (01)(23)010	12
[0101]=[010]		

[0102] $N^{(0\ 1\ 0\ 2)}$ = <*e*> has orbits {0}, {1}, {2}, {3} 12
01021 = 0(021)120 = (021)2120 and $(2120)^{\pi}$ =
0103 where π = (203) ∈ N
[01021]=[0103]

[01022]=[010]

01023 = 0(023)2031 = (023)031 and $(031)^{\pi}$ =
012 where π = (312) ∈ N
[01023]=[012]

[01020] $N^{(0\ 1\ 0\ 2\ 0)}$ = <(01)(23)> has orbits {0,1}, 6
{2,3}, 01020 ~ 10131 and for all i in {0,1}
[01020i]=[0102]

010202 = 01(02)(13)020 = (02)(13)23020 =
(02)(13)2(301)3231 = (021)23231 = (032)2321
and $(2321)^{\pi}$ = 0103 where π = (02)(13) ∈ N
[010202]=[0103]

[0103] $N^{(0\ 1\ 0\ 3)}$ = <*e*> has orbits {0}, {1}, {2}, {3} 12
01030 = (021)23202 and $(23202)^{\pi}$ = 01020
where π = (13)(20) ∈ N
[01030]=[01020]

01031 = 0(031)130 = (031)3130 and $(3130)^{\pi}$ =
0102 where π = (023) ∈ N
[01031]=[0102]

01032 = 0(320)3021 = (320)021 and $(021)^{\pi}$ =
013 where π = (213) ∈ N
[01032]=[013]

[012] $N^{(0\ 1\ 2)}$ = <*e*> has orbits {0}, {1}, {2}, {3} 12
0120 = (012)021 and $(021)^{\pi}$ = 013 where
π = (213) ∈ N
[0120]=[013]

0102 = (01)(23)01012 = (01)(23)0(123)1303
= (021)01303 = (023)0313 and $(0313)^{\pi}$ = 0121
where π = (123) ∈ N
[0121]=[0102]

[0123] $N^{(0\ 1\ 2\ 3)}$ = <(032)> has orbits {1}, {0, 2, 3} 4

0123 ~ 3102 ~ 2130 and for all j in {0,2,3}
[0123j]=[012]

01231 = 01(231)2132 = (231)0132
[01231]=[0132]

[013] $N^{(0\ 1\ 3)}$ = <e> has orbits {0}, {1}, {2}, {3} 12
0130 = (013)031 and $(031)^{\pi}$ = 012 where
π = (123) $\in$ N
[0130]=[012]

0103 = 0(132)1202 = (132)01202 = (130)0212
and $(0212)^{\pi}$ = 0131 where π = (132) $\in$ N
[0131]=[0103]

[0132] $N^{(0\ 1\ 3\ 2)}$ = <(023)> has orbits {1}, {2,3,0} 4
0132 ~ 2103 ~ 3120 and for all k in {2,3,0}
[0132k]=[013]

01321 = 01(321)3123 = (321)0123
[01321]=[0123]

The Cayley graph of $L_2(13)$ over A_4 has as vertices the set of right cosets of N in G, i.e. the set Nw_i, where w_i are words in T. The diagram below illustrates each orbit of N in its action on the vertices by right multiplication which is represented by a single node, labeled with the number of vertices that it contains. Lines in between these nodes are labeled with integers to denote how many edges from a vertex of one node will lead to the vertices of the other. At this point we can calculate the action of the four symmetric generators on the cosets of $L_2(13)$ over A_4.

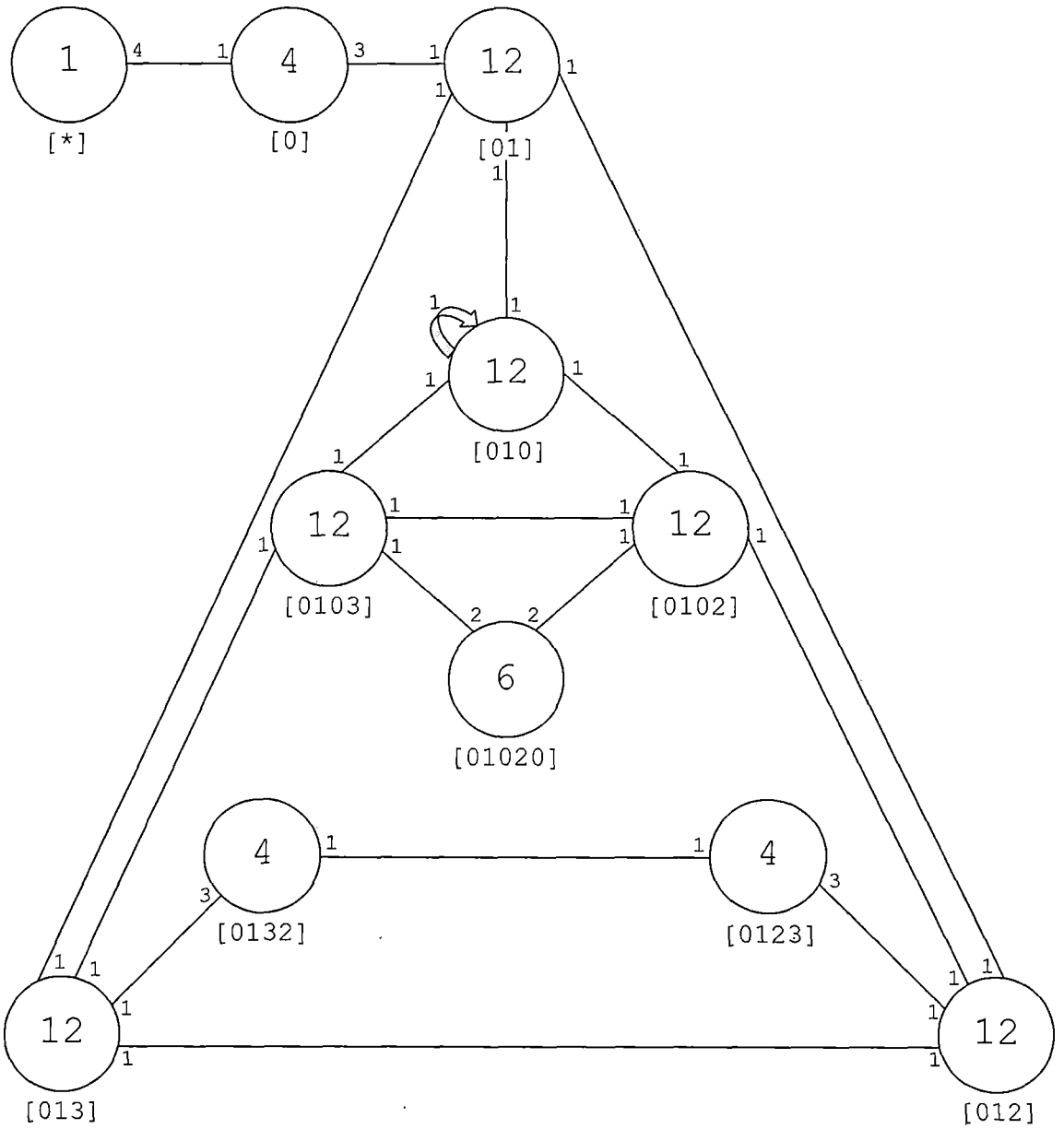


Figure 6.1 Cayley Graph of $L_2(13)$ over A_4

The labeling used for the single cosets of N in G is given by:

Label	Single Coset
1.	[N]
2.	[0]
3.	[1]
4.	[2]
5.	[1, 0]
6.	[3]
7.	[2, 0]
8.	[2, 1]
9.	[3, 0]
10.	[0, 2]
11.	[0, 1]
12.	[1, 2]
13.	[3, 2]
14.	[2, 1, 0]
15.	[1, 3]
16.	[3, 1]
17.	[0, 3]
18.	[0, 2, 0]
19.	[0, 1, 0]
20.	[2, 3]
21.	[1, 2, 0]
22.	[3, 2, 0]
23.	[0, 2, 1]
24.	[1, 3, 0]
25.	[1, 0, 2]
26.	[3, 1, 0]
27.	[0, 3, 0]
28.	[1, 0, 1]
29.	[2, 1, 2]
30.	[1, 2, 1]
31.	[2, 0, 2]
32.	[2, 3, 0]
33.	[2, 0, 1]
34.	[0, 1, 2]
35.	[3, 0, 1]
36.	[3, 1, 2]
37.	[0, 3, 2]
38.	[0, 1, 3]
39.	[2, 3, 1]
40.	[2, 0, 3]
41.	[1, 2, 1, 3]
42.	[3, 2, 1]
43.	[3, 0, 2]
44.	[1, 3, 1]
45.	[2, 3, 2]
46.	[3, 0, 3]
47.	[3, 2, 3]
48.	[2, 1, 2, 0]
49.	[1, 2, 1, 0]
50.	[3, 1, 3]
51.	[0, 3, 1]
52.	[1, 3, 2]
53.	[2, 1, 2, 3]
54.	[1, 0, 3]
55.	[0, 2, 3]
56.	[3, 1, 3, 2]
57.	[1, 2, 3]
58.	[0, 1, 3, 2]
59.	[0, 3, 2, 1]
60.	[2, 3, 2, 1]
61.	[2, 0, 2, 3]
62.	[0, 1, 0, 3]
63.	[0, 2, 3, 1]
64.	[2, 1, 3]
65.	[3, 2, 3, 1]
66.	[1, 3, 1, 0]
67.	[2, 3, 2, 0]
68.	[3, 2, 3, 0]
69.	[0, 2, 0, 1]
70.	[1, 0, 1, 2]
71.	[2, 0, 2, 1]
72.	[3, 1, 3, 0]
73.	[0, 1, 0, 2]
74.	[0, 3, 1, 2]
75.	[0, 2, 0, 3]
76.	[1, 3, 1, 2]
77.	[1, 0, 1, 3]
78.	[3, 0, 3, 1]

79. [0, 2, 1, 3]
80. [3, 0, 1, 2]
81. [0, 3, 0, 2]
82. [3, 0, 3, 2]
83. [3, 2, 3, 1, 3]
84. [3, 0, 2, 1]
85. [0, 1, 2, 3]
86. [0, 3, 0, 1]
87. [1, 3, 1, 2, 1]
88. [0, 1, 0, 2, 0]
89. [2, 0, 2, 1, 2]
90. [2, 1, 2, 3, 2]
91. [0, 3, 0, 1, 0]

The action of the symmetric generators on the cosets of N in G is given by:

t_0 = (1, 2)(3, 5)(4, 7)(6, 9)(8, 14)(10, 18)(11, 19)(12, 21)(13, 22)(15, 24)(16, 26)(17, 27)(20, 32)(23, 34)(25, 41)(29, 48)(30, 49)(33, 53)(35, 56)(36, 58)(37, 55)(38, 51)(39, 59)(40, 60)(42, 63)(43, 65)(44, 66)(45, 67)(47, 68)(50, 72)(52, 74)(54, 76)(57, 79)(61, 82)(62, 83)(64, 85)(69, 87)(70, 71)(73, 88)(75, 89)(77, 78)(80, 84)(81, 90)(86, 91)

t_1 = (1, 3)(2, 11)(4, 8)(5, 28)(6, 16)(7, 33)(9, 35)(10, 23)(12, 30)(13, 42)(14, 61)(15, 44)(17, 51)(18, 69)(20, 39)(21, 25)(22, 79)(24, 54)(26, 82)(27, 86)(31, 71)(32, 74)(34, 75)(36, 68)(37, 59)(38, 81)(40, 80)(41, 91)(43, 84)(45, 60)(46, 78)(47, 65)(48, 73)(49, 90)(52, 57)(53, 56)(55, 63)(58, 85)(62, 72)(64, 67)(66, 89)(70, 83)(76, 87)(77, 88)

t_2 = (1, 4)(2, 10)(3, 12)(5, 25)(6, 13)(7, 31)(8, 29)(9, 43)(11, 34)(14, 33)(15, 52)(16, 36)(17, 37)(19, 73)(20, 45)(21, 77)(22, 78)(23, 62)(24, 59)(26, 85)(27, 81)(28, 70)(32, 40)(35, 80)(38, 58)(39, 64)(41, 65)(42, 72)(44, 76)(46, 82)(48, 91)(49, 69)(50, 56)(51, 74)(53, 90)(54, 84)(55, 86)(57, 66)(60, 88)(61, 87)(63, 79)(67, 83)(68, 75)(71, 89)

t_3 = (1, 6)(2, 17)(3, 15)(4, 20)(5, 54)(7, 40)(8, 64)(9, 46)(10, 55)(11, 38)(12, 57)(13, 47)(14, 58)(16, 50)(18, 75)(19, 62)(21, 63)(22, 43)(23, 79)(24, 70)(25, 80)(26, 35)(28, 77)(29, 53)(30, 41)(31, 61)(32, 71)(33, 84)(34, 85)(36, 42)(37, 69)(39, 48)(49, 52)(51, 73)(56, 89)(59, 74)(60, 76)(65, 83)(66, 86)(67, 81)(68, 88)(72, 87)(78, 90)(82, 91).

Representation of Elements of $L_2(13)$

Every element of $L_2(13)$ can be represented as πw, where π is a permutation of A_4, on 4 letters, and w is a product of at most five symmetric generators.

For example:

Let α = (1, 51, 2, 16, 9, 3, 24)(4, 80, 7, 39, 32, 8, 85)
(5, 44, 27, 11, 72, 6, 78)(10, 34, 22, 65, 79, 30,
57)(12, 55, 18, 63, 68, 42, 25)(13, 84, 21, 52,
37, 23, 58)(14, 67, 60, 33, 29, 20, 31)(15, 83,
19, 81, 46, 35, 66)(17, 86, 26, 50, 76, 28, 88)
(36, 61, 41, 40, 87, 59, 73)(38, 48, 56, 82, 71,
54, 45)(43, 70, 74, 90, 64, 75, 53)(47, 77, 49,
91, 89, 69, 62),

for $\alpha \in L_2(13)$, where α is a permutation on 91 letters.

Now $N^{\alpha} = N\alpha$

$\Rightarrow$ $1^{\alpha} = 51$ (Since N is labeled as 1 and 1 goes to 51 under α)

$\Rightarrow$ $N\alpha = Nt_0t_3t_1$ (Since $Nt_0t_3t_1$ is labeled as 51)

$\Rightarrow$ $N\alpha t_1t_3t_0 = N$

$\Rightarrow$ $\alpha t_1t_3t_0 = n$, for $n \in A_4$

$\Rightarrow$ $\alpha = nt_0t_3t_1$, a symmetric representation for α.

Now, what is n?

$\alpha t_1t_3t_0$ = (1, 51, 2, 16, 9, 3, 24)(4, 80, 7, 39, 32, 8, 85)

(5, 44, 27, 11, 72, 6, 78)(10, 34, 22, 65, 79, 30, 57)(12, 55, 18, 63, 68, 42, 25)(13, 84, 21, 52, 37, 23, 58)(14, 67, 60, 33, 29, 20, 31)(15, 83, 19, 81, 46, 35, 66)(17, 86, 26, 50, 76, 28, 88) (36, 61, 41, 40, 87, 59, 73)(38, 48, 56, 82, 71, 54, 45)(43, 70, 74, 90, 64, 75, 53)(47, 77, 49, 91, 89, 69, 62) (1, 3)(2, 11)(4, 8)(5, 28)(6, 16)(7, 33)(9, 35)(10, 23)(12, 30)(13, 42)(14, 61) (15, 44)(17, 51)(18, 69)(20, 39)(21, 25)(22, 79) (24, 54)(26, 82)(27, 86)(31, 71)(32, 74)(34, 75) (36, 68)(37, 59)(38, 81)(40, 80)(41, 91)(43, 84) (45, 60)(46, 78)(47, 65)(48, 73)(49, 90)(52, 57) (53, 56)(55, 63)(58, 85)(62, 72)(64, 67)(66, 89) (70, 83)(76, 87)(77, 88) (1, 6)(2, 17)(3, 15)(4, 20)(5, 54)(7, 40)(8, 64)(9, 46)(10, 55)(11, 38) (12, 57)(13, 47)(14, 58)(16, 50)(18, 75) (19, 62) (21, 63)(22, 43)(23, 79)(24, 70)(25, 80)(26, 35) (28, 77)(29, 53)(30, 41)(31, 61)(32, 71)(33, 84) (34, 85)(36, 42)(37, 69)(39, 48)(49, 52)(51, 73) (56, 89)(59, 74)(60, 76)(65, 83)(66, 86)(67, 81) (68, 88)(72, 87)(78, 90)(82, 91) (1, 2)(3, 5)(4, 7)(6, 9)(8, 14)(10, 18)(11, 19)(12, 21)(13, 22) (15, 24)(16, 26)(17, 27)(20, 32)(23, 34)(25, 41) (29, 48)(30, 49)(33, 53)(35, 56)(36, 58)(37, 55) (38, 51)(39, 59)(40, 60)(42, 63)(43, 65)(44, 66) (45, 67)(47, 68)(50, 72)(52, 74)(54, 76)(57, 79) (61, 82)(62, 83)(64, 85)(69, 87)(70, 71)(73, 88) (75, 89)(77, 78)(80, 84)(81, 90)(86, 91)

= (Id)

Therefore, the action of $\alpha t_1 t_3 t_0$ on the symmetric generators is given by:

$Nt_1 = 3 \rightarrow 3 = Nt_1$

$Nt_2 = 4 \rightarrow 4 = Nt_2$

$Nt_3 = 6 \rightarrow 6 = Nt_3$

$Nt_0 = 2 \rightarrow 2 = Nt_0$

$\Rightarrow n = (Id)$.

And so a symmetric representation for α is given by

$\alpha = t_0t_3t_1$.

Permutation Representation of an Element Whose Symmetric Representation Is Given

Given a symmetric representation for any element of $L_2(13)$, we can easily recover its permutation representation on 91 lettters.

For example:

Let $\beta = (1\ 3\ 2)t_3$ be the symmetric representation of the element $\beta \in G$. Then the action of $\pi = (1\ 3\ 2) \in N$ on each of the ninety one single cosets of $N = A_4$ in $L_2(13)$ is given by:

Label	Single Coset	action under π	Single Coset	Label
1.	[N]	------>	[N]	= 1
2.	[0]	------>	[0]	= 2
3.	[1]	------>	[3]	= 6
4.	[2]	------>	[1]	= 3
5.	[1, 0]	------>	[3, 0]	= 9
6.	[3]	------>	[2]	= 4
7.	[2, 0]	------>	[1, 0]	= 5
8.	[2, 1]	------>	[1, 3]	= 15
9.	[3, 0]	------>	[2, 0]	= 7
10.	[0, 2]	------>	[0, 1]	= 11
11.	[0, 1]	------>	[0, 3]	= 17
12.	[1, 2]	------>	[3, 1]	= 16
13.	[3, 2]	------>	[2, 1]	= 8
14.	[2, 1, 0]	------>	[1, 3, 0]	= 24
15.	[1, 3]	------>	[3, 2]	= 13
16.	[3, 1]	------>	[2, 3]	= 20
17.	[0, 3]	------>	[0, 2]	= 10
18.	[0, 2, 0]	------>	[0, 1, 0]	= 19

19. [0, 1, 0] ------> [0, 3, 0] = 27
20. [2, 3] ------> [1, 2] = 12
21. [1, 2, 0] ------> [3, 1, 0] = 26
22. [3, 2, 0] ------> [2, 1, 0] = 14
23. [0, 2, 1] ------> [0, 1, 3] = 38
24. [1, 3, 0] ------> [3, 2, 0] = 22
25. [1, 0, 2] ------> [3, 0, 1] = 35
26. [3, 1, 0] ------> [2, 3, 0] = 32
27. [0, 3, 0] ------> [0, 2, 0] = 18
28. [1, 0, 1] ------> [3, 0, 3] = 46
29. [2, 1, 2] ------> [1, 3, 1] = 44
30. [1, 2, 1] ------> [3, 1, 3] = 50
31. [2, 0, 2] ------> [1, 0, 1] = 28
32. [2, 3, 0] ------> [1, 2, 0] = 21
33. [2, 0, 1] ------> [1, 0, 3] = 54
34. [0, 1, 2] ------> [0, 3, 1] = 51
35. [3, 0, 1] ------> [2, 0, 3] = 40
36. [3, 1, 2] ------> [2, 3, 1] = 39
37. [0, 3, 2] ------> [0, 2, 1] = 23
38. [0, 1, 3] ------> [0, 3, 2] = 37
39. [2, 3, 1] ------> [1, 2, 3] = 57
40. [2, 0, 3] ------> [1, 0, 2] = 25
41. [1, 2, 1, 3] ------> [3, 1, 3, 2] = 56
42. [3, 2, 1] ------> [2, 1, 3] = 64
43. [3, 0, 2] ------> [2, 0, 1] = 33
44. [1, 3, 1] ------> [3, 2, 3] = 47
45. [2, 3, 2] ------> [1, 2, 1] = 30
46. [3, 0, 3] ------> [2, 0, 2] = 31
47. [3, 2, 3] ------> [2, 1, 2] = 29
48. [2, 1, 2, 0] ------> [1, 3, 1, 0] = 66
49. [1, 2, 1, 0] ------> [3, 1, 3, 0] = 72
50. [3, 1, 3] ------> [2, 3, 2] = 45
51. [0, 3, 1] ------> [0, 2, 3] = 55
52. [1, 3, 2] ------> [3, 2, 1] = 42
53. [2, 1, 2, 3] ------> [1, 3, 1, 2] = 76
54. [1, 0, 3] ------> [3, 0, 2] = 43
55. [0, 2, 3] ------> [0, 1, 2] = 34
56. [3, 1, 3, 2] ------> [2, 3, 2, 1] = 60
57. [1, 2, 3] ------> [3, 1, 2] = 36
58. [0, 1, 3, 2] ------> [0, 3, 2, 1] = 59
59. [0, 3, 2, 1] ------> [0, 2, 1, 3] = 79
60. [2, 3, 2, 1] ------> [1, 2, 1, 3] = 41
61. [2, 0, 2, 3] ------> [1, 0, 1, 2] = 70
62. [0, 1, 0, 3] ------> [0, 3, 0, 2] = 81
63. [0, 2, 3, 1] ------> [0, 1, 2, 3] = 85

64. [2, 1, 3] ------> [1, 3, 2] = 52
65. [3, 2, 3, 1] ------> [2, 1, 2, 3] = 53
66. [1, 3, 1, 0] ------> [3, 2, 3, 0] = 68
67. [2, 3, 2, 0] ------> [1, 2, 1, 0] = 49
68. [3, 2, 3, 0] ------> [2, 1, 2, 0] = 48
69. [0, 2, 0, 1] ------> [0, 1, 0, 3] = 62
70. [1, 0, 1, 2] ------> [3, 0, 3, 1] = 78
71. [2, 0, 2, 1] ------> [1, 0, 1, 3] = 77
72. [3, 1, 3, 0] ------> [2, 3, 2, 0] = 67
73. [0, 1, 0, 2] ------> [0, 3, 0, 1] = 86
74. [0, 3, 1, 2] ------> [0, 2, 3, 1] = 63
75. [0, 2, 0, 3] ------> [0, 1, 0, 2] = 73
76. [1, 3, 1, 2] ------> [3, 2, 3, 1] = 65
77. [1, 0, 1, 3] ------> [3, 0, 3, 2] = 82
78. [3, 0, 3, 1] ------> [2, 0, 2, 3] = 61
79. [0, 2, 1, 3] ------> [0, 1, 3, 2] = 58
80. [3, 0, 1, 2] ------> [2, 0, 3, 1] = 80
81. [0, 3, 0, 2] ------> [0, 2, 0, 1] = 69
82. [3, 0, 3, 2] ------> [2, 0, 2, 1] = 71
83. [3, 2, 3, 1, 3] -----> [2, 1, 2, 3, 2] = 90
84. [3, 0, 2, 1] ------> [2, 0, 1, 3] = 84
85. [0, 1, 2, 3] ------> [0, 3, 1, 2] = 74
86. [0, 3, 0, 1] ------> [0, 2, 0, 3] = 75
87. [1, 3, 1, 2, 1] -----> [3, 2, 3, 1, 3] = 83
88. [0, 1, 0, 2, 0] -----> [0, 3, 0, 1, 0] = 91
89. [2, 0, 2, 1, 2] -----> [1, 0, 1, 3, 1] = 88
90. [2, 1, 2, 3, 2] -----> [1, 3, 1, 2, 1] = 87
91. [0, 3, 0, 1, 0] -----> [0, 2, 0, 3, 0] = 89

$\Rightarrow$ n = (3, 6, 4)(5, 9, 7)(8, 15, 13)(10, 11, 17)(12, 16, 20)(14, 24, 22)(18, 19, 27)(21, 26, 32)(23, 38, 37)(25, 35, 40)(28, 46, 31)(29, 44, 47)(30, 50, 45)(33, 54, 43)(34, 51, 55)(36, 39, 57)(41, 56, 60)(42, 64, 52)(48, 66, 68)(49, 72, 67)(53, 76, 65)(58, 59, 79)(61, 70, 78)(62, 81, 69)(63, 85, 74)(71, 77, 82)(73, 86, 75)(83, 90, 87)(88, 91, 89).

So, a permutation representation for β is given by

β = (3, 6, 4)(5, 9, 7)(8, 15, 13)(10, 11, 17)(12, 16, 20)
(14, 24, 22)(18, 19, 27)(21, 26, 32)(23, 38, 37)(25,
35, 40)(28, 46, 31)(29, 44, 47)(30, 50, 45)(33, 54,
43)(34, 51, 55)(36, 39, 57)(41, 56, 60)(42, 64, 52)
(48, 66, 68)(49, 72, 67)(53, 76, 65)(58, 59, 79)(61,
70, 78)(62, 81, 69)(63, 85, 74)(71, 77, 82)(73, 86,
75)(83, 90, 87)(88, 91, 89)t_3

= (3, 6, 4)(5, 9, 7)(8, 15, 13)(10, 11, 17)(12, 16, 20)
(14, 24, 22)(18, 19, 27)(21, 26, 32)(23, 38, 37)(25,
35, 40)(28, 46, 31)(29, 44, 47)(30, 50, 45)(33, 54,
43)(34, 51, 55)(36, 39, 57)(41, 56, 60)(42, 64, 52)
(48, 66, 68)(49, 72, 67)(53, 76, 65)(58, 59, 79)(61,
70, 78)(62, 81, 69)(63, 85, 74)(71, 77, 82)(73, 86,
75)(83, 90, 87)(88, 91, 89) (1, 6)(2, 17)(3, 15)(4,
20)(5, 54)(7, 40)(8, 64)(9, 46)(10, 55)(11, 38)(12,
57)(13, 47)(14, 58)(16, 50)(18, 75)(19, 62)(21, 63)
(22, 43)(23, 79)(24, 70)(25, 80)(26, 35)(28, 77)(29,
53)(30, 41)(31, 61)(32, 71)(33, 84)(34, 85)(36, 42)
(37, 69)(39, 48)(49, 52)(51, 73)(56, 89)(59, 74)(60,
76)(65, 83)(66, 86)(67, 81)(68, 88)(72, 87)(78, 90)
(82, 91)

$\Rightarrow \beta$ = (1, 6, 20, 57, 42, 8, 3)(2, 17, 55, 85, 59, 23, 11)
(4, 15, 47, 53, 60, 30, 16)(5, 46, 61, 24, 43, 84,
33)(7, 54, 22, 58, 74, 21, 35)(9, 40, 80, 25, 26,
71, 28)(10, 38, 69, 19, 27, 75, 51)(12, 50, 45, 41,
89, 68, 39)(13, 64, 49, 87, 65, 29, 44)(14, 70, 90,
72, 81, 37, 79)(18, 62, 67, 52, 36, 48, 86)(31, 77,
91, 56, 76, 83, 78)(32, 63, 34, 73, 66, 88, 82).

Proof of Isomorphism

Double coset enumeration as displayed in Figure 6.1 shows that the group defined by the symmetric presentation contains a homomorphic image of $N \cong A_4$ to index at most

$$\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(0\,1)}|} + \frac{|N|}{|N^{(0\,1\,0)}|} + \frac{|N|}{|N^{(0\,1\,2)}|} + \frac{|N|}{|N^{(0\,1\,3)}|} +$$

$$\frac{|N|}{|N^{(0\,1\,0\,2)}|} + \frac{|N|}{|N^{(0\,1\,0\,3)}|} + \frac{|N|}{|N^{(0\,1\,2\,3)}|} + \frac{|N|}{|N^{(0\,1\,3\,2)}|} + \frac{|N|}{|N^{(0\,1\,0\,2\,0)}|}$$

$$= \frac{12}{12} + \frac{12}{3} + \frac{12}{1} + \frac{12}{1} + \frac{12}{1} + \frac{12}{1} + \frac{12}{1} + \frac{12}{1} +$$

$$= \frac{12}{3} + \frac{12}{3} + \frac{12}{2}$$

$$= 1 + 4 + 12 + 12 + 12 + 12 + 12 + 12 + 4 + 4 + 6$$

$$= 91.$$

As a result, the index of N in G is at most 91.

Therefore, the order of the image group G is at most

$$91 \cdot |N| = 91 \cdot 12 = 1092.$$

Furthermore, since the action of both N and the symmetric generators on the cosets are well defined, it suffices to show that

(1) t_0 has four images, under conjugation by N

(2) the three relations hold true within the symmetric group S_{91}.

Moreover, the action by right multiplication on the 91 cosets is implicit in the enumeration, and we can easily verify that the image is the projective special linear group $L_2(13)$. In order to verify such, we need to define the linear fractional permutations of the projective line $L_2(13)$.

The projective general linear group $PGL_2(q)$ becomes the group of all operations of the form

$$x \rightarrow \frac{ax + b}{cx + d},$$

where a, b, c, d are elements of the field F_q, such that $ad - bc \neq 0$, and $q = p^n$, where p is a prime.

Furthermore, the projective special linear group $PSL_2(q)$ is a subgroup of $PGL_2(q)$. The group $PSL_2(q)$ becomes the group of all operations of the form

$$x \rightarrow \frac{ax + b}{cx + d},$$

where a, b, c, d are elements of the field F_q, such that ad - bc = 1, or equivalently with ad - bc any non-zero square in the field F_q.

And so, we define the linear fractional permutations of the projective line $L_2(13)$ as follows:

x: $\left(\tau \mapsto \frac{\tau+8}{\tau-8}\right) \equiv$ (1, 8, 14)(2, 7, 11)(4, 10, 9)(5, 13, 12)

y: $\left(\tau \mapsto \frac{8\tau+4}{\tau-4}\right) \equiv$ (1, 9, 10)(2, 3, 11)(4, 14, 8)(6, 13, 12)

t_0: $\left(\tau \mapsto \frac{11\tau+8}{\tau-11}\right) \equiv$ (1, 2)(4, 13)(5, 9)(7, 8)(10, 12)(11, 14).

We note that the order of the product xy equals 2, and so $N = \langle x, y \rangle \cong A_4$, a maximal subgroup of $L_2(13)$.

The permutation t_0 has four images under conjugation by N, namely:

t_0 = (1, 2)(4, 13)(5, 9)(7, 8)(10, 12)(11, 14)

t_1 = (1, 6)(2, 8)(3, 9)(4, 7)(5, 10)(12, 14)

t_2 = (1, 5)(3, 4)(6, 8)(7, 14)(9, 13)(10, 11)

t_3 = (1, 11)(2, 9)(3, 10)(4, 12)(6, 14)(8, 13).

As a result, the action of x and y on $\{t_0, t_1, t_2, t_3\}$ is given by:

x: (t_1, t_2, t_3)

y: (t_1, t_2, t_0).

Now, we need to verify that our relations hold true in $L_2(13)$.

Looking at our first relation we can see that

$$t_0^{(t_0t_2t_1t_0t_2t_1t_0)} = (1, 5)(3, 4)(6, 8)(7, 14)(9, 13)(10, 11)$$

$$= t_2$$

$$t_1^{(t_0t_2t_1t_0t_2t_1t_0)} = (1, 2)(4, 13)(5, 9)(7, 8)(10, 12)(11, 14)$$

$$= t_0$$

$$t_2^{(t_0t_2t_1t_0t_2t_1t_0)} = (1, 5)(3, 4)(6, 8)(7, 14)(9, 13)(10, 11)$$

$$= t_1$$

$$t_3^{(t_0t_2t_1t_0t_2t_1t_0)} = (1, 11)(2, 9)(3, 10)(4, 12)(6, 14)(8, 13)$$

$$= t_3.$$

Thus, $t_0t_2t_1t_0t_2t_1t_0$ acts as the permutation (0 2 1) on the symmetric generators, that is

$$t_0t_2t_1t_0t_2t_1t_0 = (0\ 2\ 1)$$

$$\Rightarrow t_0t_1t_2t_0 = (0\ 1\ 2)t_0t_2t_1.$$

Therefore, our first relation holds true in $L_2(13)$.

Similarly, looking at our second relation we can see that

$$t_0^{(t_0t_1t_0t_1t_0t_1t_0)} = (1,\ 6)(2,\ 8)(3,\ 9)(4,\ 7)(5,\ 10)(12,\ 14)$$

$$= t_1$$

$$t_1^{(t_0t_1t_0t_1t_0t_1t_0)} = (1,\ 2)(4,\ 13)(5,\ 9)(7,\ 8)(10,\ 12)(11,\ 14)$$

$$= t_0$$

$$t_2^{(t_0t_1t_0t_1t_0t_1t_0)} = (1,\ 11)(2,\ 9)(3,\ 10)(4,\ 12)(6,\ 14)(8,\ 13)$$

$$= t_3$$

$$t_3^{(t_0t_1t_0t_1t_0t_1t_0)} = (1,\ 5)(3,\ 4)(6,\ 8)(7,\ 14)(9,\ 13)(10,\ 11)$$

$$= t_2.$$

Thus, $t_0t_1t_0t_1t_0t_1t_0$ acts as the permutation (0 1)(2 3) on the symmetric generators, that is

$$t_0t_1t_0t_1t_0t_1t_0 = (0\ 1)(2\ 3)$$

$$\Rightarrow t_0t_1t_0t_1 = (0\ 1)(2\ 3)t_0t_1t_0.$$

Therefore, our second relation holds true in $L_2(13)$.

Looking at our third relation we can see that

$$t_0^{(t_0t_1t_2t_3t_2t_0t_1t_3)} = (1,\ 6)(2,\ 8)(3,\ 9)(4,\ 7)(5,\ 10)(12,\ 14)$$

$$= t_1$$

$$t_1^{(t_0t_1t_2t_3t_2t_0t_1t_3)} = (1,\ 5)(3,\ 4)(6,\ 8)(7,\ 14)(9,\ 13)(10,\ 11)$$

$$= t_2$$

$$t_2^{(t_0t_1t_2t_3t_2t_0t_1t_3)} = (1,\ 2)(4,\ 13)(5,\ 9)(7,\ 8)(10,\ 12)(11,\ 14)$$

$$= t_0$$

$t_3^{(t_0t_1t_2t_3t_2t_0t_1t_3)} = (1,\ 11)(2,\ 9)(3,\ 10)(4,\ 12)(6,\ 14)(8,\ 13)$

$= t_3$

Thus, $t_0t_1t_2t_3t_2t_0t_1t_3$ acts as the permutation (0 1 2) on the symmetric generators, that is

$$t_0t_1t_2t_3t_2t_0t_1t_3 = (0\ 1\ 2)$$

$$\Rightarrow t_0t_1t_2t_3 = (0\ 1\ 2)t_3t_1t_0t_2.$$

Therefore, our third relation holds true in $L_2(13)$.

Since our relations hold in $L_2(13)$, N is maximal in $L_2(13)$ and $t_0 \notin N$, where t_0 has 4 conjugates under conjugation by N, then $L_2(13)$ is generated by N and t_0. Therefore, $L_2(13)$ is an image of G. As a result, $|G| \geq |L_2(13)| = 1092$. But from previous we know that $|G| \leq 1092$, and so

$$|G| \leq 1092 = |L_2(13)| \leq |G|,$$

which proves the isomorphism. Hence $G \cong L_2(13)$.

CHAPTER SEVEN

SYMMETRIC REPRESENTAION OF $L_2(41)$ ELEMENTS

Introduction

In this chapter we will show that the group $L_2(41)$ is generated by 6 involutions which are permuted, by conjugation within $L_2(41)$, by a subgroup isomorphic to $L_2(5)$, and that each element of $L_2(41)$ can be written (not necessarily uniquely) as a permutation of $L_2(5)$ on 6 letters followed by a word of length less than or equal to five in these generators.

Factoring the progenitor $2^{*6} : L_2(5)$ by the following relations

$$[(0\ 1\ 2\ 3\ 4)t_0]^7,\ [(\infty\ 0\ 1)(2\ 4\ 3)t_\infty]^7,$$

$$t_0t_1t_2t_0 = (0\ \infty\ 2\ 4\ 3)t_4t_0t_1t_4,\ t_0t_1t_2t_\infty t_3 = (2\ 4)(3\ \infty)t_3t_\infty t_1t_2t_4,$$

$$t_0t_1t_2t_\infty t_4 = (1\ 4)(2\ 3)t_4t_3t_0t_1t_2,\ [(0\ 3\ 2)(1\ \infty\ 4)t_0t_1t_0t_1t_0]^2$$

yields the finite homomorphic image:

$$G \cong \frac{2^{*6} : L_2(5)}{\begin{array}{l}[(0\ 1\ 2\ 3\ 4)t_0]^7,\ [(\infty\ 0\ 1)(2\ 4\ 3)t_\infty]^7,\\ t_0t_1t_2t_0 = (0\ \infty\ 2\ 4\ 3)t_4t_0t_1t_4,\ t_0t_1t_2t_\infty t_3 = (2\ 4)(3\ \infty)t_3t_\infty t_1t_2t_4\\ [(0\ 3\ 2)(1\ \infty\ 4)t_0t_1t_0t_1t_0]^2,\ t_0t_1t_2t_\infty t_4 = (1\ 4)(2\ 3)t_4t_3t_0t_1t_2\end{array}},$$

where $L_2(5)$ is maximal in $L_2(41)$ and the index of $L_2(5)$ in G equals 574. $G \cong L_2(41)$, the projective special linear group.

A symmetric presentation for the above image is given by:

<x,y,t|x^5, y^3, (x*y)^2, t^2, (t,(y*x)^(x^2*y^2)),
(t,(x^3)^((y*x)^(y^2*x^3*y))), (x*t)^7, (y*t^(y^2))^7,
t*t^x*t^(x^2)*t=(x^(y^2))^(-1)*t^(x^4)*t*t^x*t^(x^4),
t*t^x*t^(x^2)*t^(y^2)*t^(x^3)=(x*y)^(x^3)*t^(x^3)*
t^(y^2)*t^x*t^(x^2)*t^(x^4),t*t^x*t^(x^2)*t^(y^2)*
t^(x^4)=(x*y)^(x^3*y^2)*t^(x^4)*t^(x^3)* t*t^x*
t^(x^2),(y^(x*y^2*x^2)*t*t^x*t*t^x*t)^2>,

where $N = L_2(5) \cong$ <x,y|x^5, y^3, (x*y)^2>,

and the action of x, y on the symmetric generators is given by

$$x \sim (0\ 1\ 2\ 3\ 4)$$

$$y \sim (\infty\ 0\ 1)(2\ 4\ 3).$$

Double Coset Enumeration

The double coset [0] has 6 distinct single cosets, namely Nt_0, Nt_1, Nt_2, Nt_3, Nt_4, and Nt_∞. Meanwhile, the double coset [0 1] contains 30 distinct single cosets, where $N^{(0\ 1)}$ has orbits {0}, {1}, {2, 4}, and {3, ∞}.

The double coset [0 1 2] has 60 distinct single cosets.

Now, expanding the relation $[(0\ 1\ 2\ 3\ 4)t_0]^7 = 1$ yields:

$(\pi t_0)^7 = 1$, where $\pi = (0\ 1\ 2\ 3\ 4) \in N$

$$\Rightarrow \pi^7(t_0)^{\pi^6}(t_0)^{\pi^5}(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}(t_0) = 1$$

$\Rightarrow$ $(1\ 3\ 0\ 2\ 4)t_1t_0t_4t_3t_2t_1t_0 = 1$

$\Rightarrow$ $\boxed{t_0t_1t_2t_3 = (1\ 3\ 0\ 2\ 4)t_1t_0t_4 \qquad (1)}$

$\Rightarrow$ $Nt_0t_1t_2t_3 = Nt_1t_0t_4$,

but $Nt_1t_0t_4 \in [0\ 1\ 2]$ (since $(Nt_1t_0t_4)^{\pi} = Nt_0t_1t_2$,

where $\pi = (0\ 1)(2\ 4) \in N$).

Therefore, $[0\ 1\ 2\ 3] = [0\ 1\ 2]$.

Similarly, $[0\ 1\ 2\ 1] = [0\ 1\ 0\ 2]$.

Now, looking at the relation $t_0t_1t_2t_0 = (0\ \infty\ 2\ 4\ 3)t_4t_0t_1t_4$ we have:

$t_0t_1t_2t_0 = (0\ \infty\ 2\ 4\ 3)t_4t_0t_1t_4$

$\Rightarrow$ $Nt_0t_1t_2t_0 = Nt_4t_0t_1t_4$

$\Rightarrow$ $(Nt_0t_1t_2t_0)^{(1\ 2\ 3\ 4\ 0)} = (Nt_4t_0t_1t_4)^{(1\ 2\ 3\ 4\ 0)}$

$\Rightarrow$ $Nt_1t_2t_3t_1 = Nt_0t_1t_2t_0$

$\Rightarrow$ $(Nt_1t_2t_3t_1)^{(1\ 2\ 3\ 4\ 0)} = (Nt_0t_1t_2t_0)^{(0\ 1\ 2\ 3\ 4\ 0)}$

$\Rightarrow$ $Nt_2t_3t_4t_2 = Nt_1t_2t_3t_1$

$\Rightarrow$ $(Nt_2t_3t_4t_2)^{(1\ 2\ 3\ 4\ 0)} = (Nt_1t_2t_3t_1)^{(1\ 2\ 3\ 4\ 0)}$

$\Rightarrow$ $Nt_3t_4t_0t_3 = Nt_2t_3t_4t_2$.

Therefore, $Nt_0t_1t_2t_0 = Nt_4t_0t_1t_4 = Nt_1t_2t_3t_1 = Nt_2t_3t_4t_2 = Nt_3t_4t_0t_3$.

The double coset $[0\ 1\ 2\ 0]$ contains $5 \cdot 4 \cdot 3 \cdot 1 = 60$ single cosets. But since $Nt_0t_1t_2t_0$, has five names, then the number of distinct single cosets in $[0\ 1\ 2\ 0]$ turns out to

be $\frac{60}{5} = 12$.

Now, applying t_1 to $Nt_0t_1t_2t_0 = Nt_1t_2t_3t_1$ yields:

$Nt_0t_1t_2t_0t_1 = Nt_1t_2t_3t_1t_1$

$= Nt_1t_2t_3,$

but $Nt_1t_2t_3 \in [0\ 1\ 2]$ (since $(Nt_1t_2t_3)^{\pi} = Nt_0t_1t_2,$

where $\pi = (1\ 0\ 4\ 3\ 2) \in N$).

Therefore, $[0\ 1\ 2\ 0\ 1] = [0\ 1\ 2]$.

However, since $N^{(0\ 1\ 2\ 0)}$ has orbits $\{0, 1, 2, 3, 4\}$ and $\{\infty\}$,

then $[0\ 1\ 2\ 0\ i] = [0\ 1\ 2]$ for all i in $\{0, 1, 2, 3, 4\}$.

Now, $Nt_0t_1t_2t_0t_\infty$ may be written as

$Nt_0t_1t_2t_0t_\infty = Nt_0(2\ \infty\ 1\ 0\ 3)t_2t_1t_3$

$= Nt_3t_2t_1t_3,$

but $Nt_3t_2t_1t_3 \in [0\ 1\ 2\ 0]$ (since $(Nt_3t_2t_1t_3)^{\pi} = Nt_0t_1t_2t_0,$

where $\pi = (1\ 2)(3\ 0) \in N$).

Therefore, $[0\ 1\ 2\ 0\ \infty] = [0\ 1\ 2\ 0]$.

The double coset $[0\ 1\ 2\ 4]$ contains 60 single cosets

Now, $Nt_0t_1t_2t_4t_1$ may be written as

$Nt_0t_1t_2t_4t_1 = Nt_0(2\ 4\ 1)(\infty\ 0\ 3)t_1t_4t_2$

$= Nt_3t_1t_4t_2,$

but $Nt_3t_1t_4t_2 \in [0\ 1\ 3\ 4]$ (since $(Nt_3t_1t_4t_2)^{\pi} = Nt_0t_1t_3t_4,$

where $\pi = (2\ 4\ 3\ 0\ \infty) \in N$).

Therefore, $[0\ 1\ 2\ 4\ 1] = [0\ 1\ 3\ 4]$.

Now, $t_0t_1t_2t_4t_2$ can be written as

$t_0t_1t_2t_4t_2 = t_0(1\ 3\ \infty)(2\ 0\ 4)t_4t_0t_4t_\infty$

$= (1\ 3\ \infty)(2\ 0\ 4)t_4t_4t_0t_4t_\infty$

$= (1\ 3\ \infty)(2\ 0\ 4)t_0t_4t_\infty,$

but $t_0t_4t_\infty \in [0\ 1\ 3]$ (since $(t_0t_4t_\infty)^\pi = t_0t_1t_3$,

where $\pi = (1\ 2\ \infty\ 3\ 4) \in N$).

Therefore, $[0\ 1\ 2\ 4\ 2] = [0\ 1\ 3]$. Similarly, $[0\ 1\ 2\ 4\ 3] = [0\ 1\ 2\ \infty\ 4]$, $[0\ 1\ 2\ 4\ \infty] = [0\ 1\ 0\ 1]$, and $[0\ 1\ 2\ 4\ 0] = [0\ 1\ 0\ 2]$.

The double coset $[0\ 1\ 2\ \infty]$ has 60 single cosets.

Now, $Nt_0t_1t_2t_\infty t_1$ may be written as

$Nt_0t_1t_2t_\infty t_1 = Nt_0(2\ \infty\ 1)(4\ 3\ 0)t_1t_\infty t_2$

$= Nt_4t_1t_\infty t_2,$

but $Nt_4t_1t_\infty t_2 \in [0\ 1\ 2\ \infty]$ (since $Nt_4t_1t_\infty t_2 = Nt_0t_1t_2t_\infty$,

where $\pi = (2\ \infty)(4\ 0) \in N$).

Therefore, $[0\ 1\ 2\ \infty\ 1] = [0\ 1\ 2\ \infty]$.

Now, $t_0t_1t_2t_\infty t_0$ may be written as

$t_0t_1t_2t_\infty t_0 = t_0(1\ \infty\ 4\ 2\ 3)t_0t_3t_\infty t_2$

$= (1\ \infty\ 4\ 2\ 3)t_0t_0t_3t_\infty t_2$

$= (1\ \infty\ 4\ 2\ 3)t_3t_\infty t_2,$

but $t_3t_\infty t_2 \in [0\ 1\ 3]$ (since $(t_3t_\infty t_2)^\pi = t_0t_1t_3$,

where $\pi = (1\ 4\ \infty)(2\ 3\ 0) \in N$).

Therefore, [0 1 2 ∞ 0] = [0 1 3]. Similarly, [0 1 2 ∞ 2] = [0 1 3 4].

Now, looking at the relation $t_0t_1t_2t_\infty t_3 = (2\ 4)(3\ \infty)t_3t_\infty t_1t_2t_4$ we have:

$t_0t_1t_2t_\infty t_3 = (2\ 4)(3\ \infty)t_3t_\infty t_1t_2t_4$

$\Rightarrow Nt_0t_1t_2t_\infty t_3 = Nt_3t_\infty t_1t_2t_4$

$\Rightarrow (Nt_0t_1t_2t_\infty t_3)^{(1\ \infty\ 2)(3\ 4\ 0)} = (Nt_3t_\infty t_1t_2t_4)^{(1\ \infty\ 2)(3\ 4\ 0)}$

$\Rightarrow Nt_3t_\infty t_1t_2t_4 = Nt_4t_2t_6t_1t_0.$

Therefore, $Nt_0t_1t_2t_\infty t_3 = Nt_3t_\infty t_1t_2t_4 = Nt_4t_2t_\infty t_1t_0.$

As a result, double coset [0 1 2 ∞ 3] contains 20 distinct single cosets, each with three names.

Now, applying t_0 to $Nt_0t_1t_2t_\infty t_3 = Nt_4t_2t_\infty t_1t_0$ yields:

$Nt_0t_1t_2t_\infty t_3t_0 = Nt_4t_2t_\infty t_1t_0t_0$

$= Nt_4t_2t_\infty t_1,$

but $Nt_4t_2t_\infty t_1 \in$ [0 1 2 ∞] (since $(Nt_4t_2t_\infty t_1)^\pi = Nt_0t_1t_2t_\infty$, where $\pi = (1\ \infty\ 2)(3\ 4\ 0) \in N$).

Therefore, [0 1 2 ∞ 3 0] = [0 1 2 ∞].

However, since $N^{(0\ 1\ 2\ \infty\ 3)}$ has orbits {1, 2, ∞} and {0, 3, 4}, then [0 1 2 ∞ 3 j] = [0 1 2 6] for all j in {0, 3, 4}.

Similarly, [0 1 2 ∞ 3 k] = [0 1 3 4] for all k in {1, 2, ∞}.

Now, looking at the relation $t_0t_1t_2t_\infty t_4 = (1\ 4)(2\ 3)t_4t_3t_0t_1t_2$ we have:

$t_0t_1t_2t_\infty t_4 = (1\ 4)(2\ 3)t_4t_3t_0t_1t_2$

$\Rightarrow Nt_0t_1t_2t_\infty t_4 = Nt_4t_3t_0t_1t_2$

$\Rightarrow (Nt_0t_1t_2t_\infty t_4)^{(1\ \infty\ 3)(2\ 4\ 0)} = (Nt_4t_3t_0t_1t_2)^{(1\ \infty\ 3)(2\ 4\ 0)}$

$\Rightarrow Nt_2t_\infty t_4t_3t_0 = Nt_0t_1t_2t_\infty t_4.$

Therefore, $Nt_0t_1t_2t_\infty t_4 = Nt_4t_3t_0t_1t_2 = Nt_2t_\infty t_4t_3t_0.$

As a result, double coset [0 1 2 ∞ 4] contains 20 distinct single cosets, each with three names.

Now, applying t_0 to $Nt_0t_1t_2t_\infty t_4 = Nt_2t_\infty t_4t_3t_0$ yields:

$Nt_0t_1t_2t_\infty t_4t_0 = Nt_2t_\infty t_4t_3t_0t_0$

$= Nt_2t_\infty t_4t_3,$

but $Nt_2t_\infty t_4t_3 \in [0\ 1\ 2\ \infty]$ (since $(Nt_2t_\infty t_4t_3)^\pi = Nt_0t_1t_2t_\infty$, where $\pi = (1\ 3\ \infty)(2\ 0\ 4) \in N$).

Therefore, [0 1 2 ∞ 4 0] = [0 1 2 ∞].

However, since $N^{(0\ 1\ 2\ \infty\ 4)}$ has orbits {1, 3, ∞} and {2, 4, 0}, then [0 1 2 ∞ 4 m] = [0 1 2 ∞] for all m in {0, 2, 4}.

Now, $t_0t_1t_2t_\infty t_4t_1 = t_0t_1(\infty\ 1\ 2\ 4\ 0)t_\infty t_2t_0$

$= (\infty\ 1\ 2\ 4\ 0)t_\infty t_2t_\infty t_2t_0$

$= (1\ 2\ 0\ \infty\ 3)t_4t_2t_3t_\infty,$

but $t_4t_2t_3t_\infty \in [0\ 1\ 2\ 4]$ (since $(t_4t_2t_3t_\infty)^\pi = t_0t_1t_2t_4$, where $\pi = (1\ 3\ 2)(4\ 5\ 0) \in N$).

Therefore, [0 1 2 ∞ 4 1] = [0 1 2 4], which implies that

[0 1 2 ∞ 4 p] = [0 1 2 4] for all p in {1, 3, ∞}.

The double coset [0 1 3] has 60 single cosets.

Now, expanding the relation $[(\infty\ 0\ 1)(2\ 4\ 3)t_\infty]^7 = 1$ yields:

$(\pi t_\infty)^7 = 1$, where $\pi = (\infty\ 0\ 1)(2\ 4\ 3)$

$\Rightarrow \pi^7(t_\infty)^{\pi^6}(t_\infty)^{\pi^5}(t_\infty)^{\pi^4}(t_\infty)^{\pi^3}(t_\infty)^{\pi^2}(t_\infty)^{\pi}(t_\infty) = 1$

$\Rightarrow (1\ \infty\ 0)(2\ 4\ 3)t_\infty t_1 t_0 t_\infty t_1 t_0 t_\infty = 1$

$\Rightarrow \boxed{(1\ \infty\ 0)(2\ 4\ 3)t_\infty t_1 t_0 t_\infty = t_\infty t_0 t_1 \quad (2)}$

$\Rightarrow Nt_\infty t_1 t_0 t_\infty = Nt_\infty t_0 t_1$.

Conjugating (2) by $(2\ \infty\ 0\ 3\ 4) \in N$

$\Rightarrow (1\ 0\ 3)(\infty\ 2\ 4)t_0 t_1 t_3 t_0 = t_0 t_3 t_1$

$\Rightarrow Nt_0 t_1 t_3 t_0 = Nt_0 t_3 t_1$,

but $Nt_0 t_3 t_1 \in [0\ 1\ 3]$ (since $(Nt_0 t_3 t_1)^\pi = Nt_0 t_1 t_3$, where where $\pi = (1\ 3)(2\ \infty) \in N$).

Therefore, [0 1 3 0] = [0 1 3].

Similarly, [0 1 3 1] = [0 1 0 3], [0 1 3 2] = [0 1 2 ∞],

and [0 1 3 ∞] = [0 1 2 4].

The double coset [0 1 3 4] has 60 single cosets.

$Nt_0 t_1 t_3 t_4 t_1$ may be written as

$Nt_0 t_1 t_3 t_4 t_1 = Nt_0(3\ 4\ 1)(0\ \infty\ 2)t_1 t_4 t_3$

$= Nt_\infty t_1 t_4 t_3$,

but $Nt_\infty t_1 t_4 t_3 \in [0\ 1\ 2\ 4]$ (since $(Nt_\infty t_1 t_4 t_3)^\pi = Nt_0 t_1 t_2 t_4$,

where $\pi = (2\ \infty\ 0\ 3\ 4) \in N$).

Therefore, [0 1 3 4 1] = [0 1 2 4].

Now, $t_0t_1t_3t_4t_0$ may be written as

$t_0t_1t_3t_4t_0 = t_0(1\ 0\ 3)(\infty\ 2\ 4)t_1t_2t_0t_3$

$= (1\ 0\ 3)(\infty\ 2\ 4)t_3t_1t_2t_0t_3$

$= (1\ 3\ 0\ 2\ 4)t_1t_3t_\infty t_3$

$= (1\ 6\ 2)(3\ 4\ 0)t_2t_3t_2t_4$,

but $t_2t_3t_4t_2 \in$ [0 1 0 2] (since $(t_2t_3t_4t_2)^\pi = t_0t_1t_0t_2$,

where $\pi = (1\ 4\ 2\ 0\ 3) \in N$).

Therefore, [0 1 3 4 0] = [0 1 0 2].

Now, $t_0t_1t_3t_4t_3$ may be written as

$t_0t_1t_3t_4t_3 = t_0(1\ 2\ 0)(3\ \infty\ 4)t_4t_\infty t_4t_0$

$= (1\ 2\ 0)(3\ \infty\ 4)t_1t_4t_\infty t_4t_0$

$= (1\ 3\ 0\ 2\ 4)t_0t_4t_0t_3t_0$

$= (1\ 2\ 4)(3\ \infty\ 0)t_0t_\infty t_0t_4$,

but $t_0t_\infty t_0t_4 \in$ [0 1 0 3] (since $(t_0t_\infty t_0t_4)^\pi = t_0t_1t_0t_3$,

where $\pi = (1\ \infty)(3\ 4) \in N$).

Therefore, [0 1 3 4 3] = [0 1 0 3].

Similarly, [0 1 3 4 2] = [0 1 2 ∞ 3] and [0 1 3 4 ∞] = [0 1 2 ∞].

The double coset [0 1 0] has 30 single cosets, where $N^{(0\ 1\ 0)}$ has orbits {0}, {1}, {2, 4}, and {3, ∞}. Meanwhile, the double coset [0 1 0 2] has 60 single cosets.

Now, $t_0t_1t_0t_2t_1$ can be written as

$t_0t_1t_0t_2t_1 = t_0(1\ 3\ 2\ 4\ \infty)t_4t_1t_0t_4$

$= (1\ 3\ 2\ 4\ \infty)t_0t_4t_1t_0t_4$

$= (1\ \infty\ 3\ 0\ 4)t_4t_0t_2t_\infty t_0$

$= (1\ 0\ \infty)(2\ 3\ 4)t_4t_1t_\infty t_0t_0$

$= (1\ 0\ \infty)(2\ 3\ 4)t_4t_1t_\infty,$

but $t_4t_1t_\infty \in [0\ 1\ 2]$ (since $(t_4t_1t_\infty)^\pi = t_0t_1t_2$,

where $\pi = (2\ \infty)(0\ 4) \in N$).

Therefore, $[0\ 1\ 0\ 2\ 1] = [0\ 1\ 2]$.

Similarly, $[0\ 1\ 0\ 2\ 3] = [0\ 1\ 3\ 4]$ and $[0\ 1\ 0\ 2\ 4] =$

$[0\ 1\ 0\ 1]$.

Now, $t_0t_1t_0t_2t_\infty$ can be written as

$t_0t_1t_0t_2t_\infty = t_0(0\ \infty\ 1\ 2\ 4)t_0t_1t_4$

$= (0\ \infty\ 1\ 2\ 4)t_\infty t_0t_1t_4$

$= (2\ 0\ 4\ \infty\ 3)t_\infty t_2t_4t_0,$

but $t_\infty t_2t_4t_0 \in [0\ 1\ 2\ 4]$ (since $(t_\infty t_2t_4t_0)^\pi = t_0t_1t_2t_4$,

where $\pi = (1\ \infty\ 0\ 4\ 2) \in N$).

Therefore, $[0\ 1\ 0\ 2\ \infty] = [0\ 1\ 2\ 4]$.

Similarly, $[0\ 1\ 0\ 2\ 0] = [0\ 1\ 0\ 3]$.

The double coset $[0\ 1\ 0\ 3]$ has 60 single cosets.

Now, $Nt_0t_1t_0t_3t_1$ may be written as

$Nt_0t_1t_0t_3t_1 = Nt_0(3\ 1\ 0)(2\ 4\ \infty)t_1t_3t_0$

$= Nt_3t_1t_3t_0,$

but $Nt_3t_1t_3t_0 \in [0\ 1\ 0\ 3]$ (since $(Nt_3t_1t_3t_0)^{\pi} = Nt_0t_1t_0t_3$, where $\pi = (3\ 0)(4\ \infty) \in N$).

Therefore, [0 1 0 3 1] = [0 1 0 3]. Similarly, [0 1 0 3 2] = [0 1 0 3], [0 1 0 3 ∞] = [0 1 3], [0 1 0 3 0] = [0 1 0 2], and [0 1 0 3 4] = [0 1 3 4].

The double coset [0 1 0 1] has 30 single cosets, where [0 1 0 1 2] = [0 1 2 4], and [0 1 0 1 3] = [0 1 0 2]. However, since $N^{(0\ 1\ 0\ 1)}$ has orbits {0}, {1}, {2, 4}, and {3, ∞}, then [0 1 0 1 q] = [0 1 2 4] for all q in {2, 4}, and [0 1 0 1 r] = [0 1 0 2] for all r in {3, ∞}.

Now, expanding the relation $[(0\ 3\ 2)(1\ \infty\ 4)t_0t_1t_0t_1t_0]^2 = 1$ yields:

$\pi^2(t_0t_1t_0t_1t_0)^{\pi}(t_0t_1t_0t_1t_0) = 1$, where $\pi = (0\ 3\ 2)(1\ \infty\ 4)$

$\Rightarrow (0\ 2\ 3)(1\ 4\ \infty)t_3t_\infty t_3t_\infty t_3t_0t_1t_0t_1t_0 = 1$

$\Rightarrow t_0t_1t_0t_1t_0 = (0\ 2\ 3)(1\ 4\ \infty)t_3t_\infty t_3t_\infty t_3$

$\Rightarrow Nt_0t_1t_0t_1t_0 = Nt_3t_\infty t_3t_\infty t_3$

$\Rightarrow (Nt_0t_1t_0t_1t_0)^{(1\ 4\ \infty)(2\ 3\ 0)} = (Nt_3t_\infty t_3t_\infty t_3)^{(1\ 4\ \infty)(2\ 3\ 0)}$

$\Rightarrow Nt_2t_4t_2t_4t_2 = Nt_0t_1t_0t_1t_0$

$\Rightarrow (Nt_2t_4t_2t_4t_2)^{(1\ 4\ 3)(2\ \infty\ 0)} = (Nt_0t_1t_0t_1t_0)^{(1\ 4\ 3)(2\ \infty\ 0)}$

$\Rightarrow Nt_\infty t_3t_\infty t_3t_\infty = Nt_2t_4t_2t_4t_2$

$\Rightarrow (Nt_\infty t_3t_\infty t_3t_\infty)^{(1\ 4\ \infty)(2\ 3\ 0)} = (Nt_2t_4t_2t_4t_2)^{(1\ 4\ \infty)(2\ 3\ 0)}$

$\Rightarrow Nt_1t_0t_1t_0t_1 = Nt_3t_\infty t_3t_\infty t_3$

$\Rightarrow (Nt_1t_0t_1t_0t_1)^{(1\ 4\ \infty)(2\ 3\ 0)} = (Nt_3t_\infty t_3t_\infty t_3)^{(1\ 4\ \infty)(2\ 3\ 0)}$

$\Rightarrow Nt_4t_2t_4t_2t_4 = Nt_0t_1t_0t_1t_0$

Therefore, $Nt_0t_1t_0t_1t_0 = Nt_2t_4t_2t_4t_2 = Nt_3t_\infty t_3t_\infty t_3 = Nt_\infty t_3t_\infty t_3t_\infty =$

$Nt_1t_0t_1t_0t_1 = Nt_4t_2t_4t_2t_4.$

As a result, the double coset [0 1 0 1 0] contains 5 distinct single cosets, each with six names.

Applying t_1 to $Nt_0t_1t_0t_1t_0 = Nt_1t_0t_1t_0t_1$ yields:

$Nt_0t_1t_0t_1t_0t_1 = Nt_1t_0t_1t_0t_1t_1$

$= Nt_1t_0t_1t_0,$

but $Nt_1t_0t_1t_0 \in$ [0 1 0 1] (since $(Nt_1t_0t_1t_0)^\pi = Nt_0t_1t_0t_1$, where $\pi = (0\ 1)(2\ 4) \in N$).

Therefore, [0 1 0 1 0 1] = [0 1 0 1].

However, since $N^{(0\ 1\ 0\ 1\ 0)}$ has orbit {0, 1, 2, 3, 4, ∞} then [0 1 0 1 0 s] = [0 1 0 1] for all s in {0, 1, 2, 3, 4, ∞}.

The set of all double cosets [w] = NwN, coset stabilizing subgroups $N^{(w)}$, and the number of single cosets are exhibited in Table 7.1. The table shows that the Cayley Graph of G over N has the form as displayed in Figure 7.1.

Table 7.1 The Double Cosets [w] = NwN in G, Where N = $L_2(5)$

NwN	Reason	Number of single cosets
[*]	N is transitive	1
[0]	$N^{(0)} = \langle (12)(4\infty), (14)(23) \rangle$ has orbits {0}, {1, 2, 3, 4, 5, 6}	6

[00]=[*]

[01] $N^{(0\ 1)}$ = <(24)(3∞)> has orbits {0}, {1}, {2, 4}, {3, ∞}. 30

[012] $N^{(0\ 1\ 2)}$ = <*e*> has orbits {0}, {1}, {2}, {3}, {4}, {∞}, 0123 = (13024)104 and $(104)^{\pi}$ = 012 where π = (01)(24) ∈ N 60

[0123]=[012]

0121 = (03∞)(124)414∞ and $(414\infty)^{\pi}$ = 0102 where π = (2∞)(40) ∈ N

[0121]=[0102]

[0120] $N^{(0\ 1\ 2\ 0)}$ = <(12340)> has orbits {∞}, {0 1, 2, 3, 4}, 0120 ~ 4014 ~ 1231 ~ 2342 ~ 3403 and for all i in {0, 1, 2, 3, 4} 12

[0120i]=[012]

0120∞ = 0(2∞)(03)213 = (2∞103)3213 and $(3213)^{\pi}$ = 0120 where π = (12)(30 ∈ N

[0120∞]=[0120]

[0124] $N^{(0\ 1\ 2\ 4)}$ = <*e*> has orbits {0}, {1}, {2}, {3}, {4}, {∞}, 01241 = 0(241)(∞03)142 = (241)(∞03)3142 and $(3142)^{\pi}$ = 0134 where π = (2430∞) ∈ N 60

[01241]=[0134]

01242 = 0(13∞)(204)404∞ = (13∞)(204)04∞ and $(04\infty)^{\pi}$ = 013 where π = (12∞34) ∈ N

[01242]=[013]

01243 = (1324∞)∞3140 and $(\infty3140)^{\pi}$ = 012∞4 where π = (123)(04∞) ∈ N

[01243]=[012∞4]

0124∞ = (0∞)(23)4141 and $(4141)^{\pi}$ = 0101 where π = (∞2)(04) ∈ N

[0124∞]=[0101]

01240 = (02413)4∞41 and $(4\infty41)^{\pi}$ = 0102 where π = (1240∞) ∈ N

[01240]=[0102]

[012∞] $N^{(0\ 1\ 2\ \infty)}$ = <*e*> has orbits {0}, {1}, {2}, {3}, {4}, {∞}, 012∞1 = 0(2∞1)(430)1∞2 = (2∞1)(43)41∞2 and $(41\infty2)^{\pi}$ = 012∞ where π = (2∞)(40) ∈ N 60

[012∞1]=[012∞]

012∞0 = 0(1∞423)03∞2 = (1∞423)3∞2 and $(3\infty2)^{\pi}$ = 013 where π = (14∞)(230) ∈ N

[012∞0]=[013]

012∞2 = (0∞2)(134)3∞40 and $(3\infty40)^{\pi}$ = 0134 where π = (12∞)(304) ∈ N

[012∞2]=[0134]

[012∞3] $N^{(0\ 1\ 2\ \infty\ 3)}$ = <(1∞2)(340)> has orbits {1, 2, ∞}, {0, 3, 4}, 012∞3 ~ 3∞124 ~ 42∞10 and for all j in {0, 3, 4} 20

[0123j]=[012∞]

012∞31 = (0421∞)3204 and $(3204)^{\pi}$ = 0134 where π = (12)(30) ∈ N

[012∞31]=[0134]

[012∞4] $N^{(0\ 1\ 2\ \infty\ 4)}$ = <(13∞)(204)> has orbits {1, 3, ∞}, {0, 2, 4}, 012∞4 ~ 43012 ~ 2∞430 and for all k in {0, 2, 4} 20

[012∞4k]=[012∞]

012∞41 = 01(∞1240)∞20 = (∞1240)∞2∞20 = (120∞3)423∞ and $(423\infty)^{\pi}$ = 0124 where π = (132)(40∞) ∈ N

[012∞41]=[0124]

[013] $N^{(0\ 1\ 3)}$ = <*e*> has orbits {0}, {1}, {2}, {3}, {4}, {∞}, 0130 = (130)(∞42)031 and $(031)^{\pi}$ = 013 where π = (13)(2∞) ∈ N 60

[0130]=[013]

0132 = (03∞14)2431 and $(2431)^{\pi}$ = 012∞ where π = (1∞4)(203) ∈ N

[0132]=[012∞]

013∞ = (0∞1)(423)02∞1 and $(02\infty1)^{\pi}$ = 0124 where π = (143∞2) ∈ N

[013∞]=[0124]

[0134] $N^{(0\ 1\ 3\ 4)}$ = <*e*> has orbits {0}, {1}, {2}, {3}, {4}, {∞}, 01341 = 0(341)(0∞2)143 = (341)(0∞2)∞143 and $(\infty 143)^{\pi}$ = 0124 where π = (2∞034) ∈ N 60

[01341]=[0124]

01340 = 0(103)(∞24)1203 = (103)(∞24)31203 = (13024)13∞3 = (1∞2)(340)2324 and $(2324)^{\pi}$ = 0102 where π = (14203) ∈ N

[01340]=[0102]

01343 = 0(120)(3∞4)4∞40 = (120)(3∞4)14∞40 = (13024)04030 = (124)(3∞0)0∞04 and $(0\infty 04)^{\pi}$ = 0103 where π = (1∞)(34) ∈ N

[01343]=[0103]

01342 = (0∞243)0∞214 and $(0\infty 214)^{\pi}$ = 012∞3 where π = (1∞)(24) ∈ N

[01342]=[012∞3]

0134∞ = (023)(14∞)42∞1 and $(42\infty 1)^{\pi}$ = 012∞ where π = (1∞2)(340) ∈ N

[0134∞]=[0120]

[010] $N^{(0\ 1\ 0)}$ = <(24)(3∞)> has orbits {0}, {1}, {2, 4}, {3, ∞} 30

[0102] $N^{(0\ 1\ 0\ 2)}$ = <*e*> has orbits {0}, {1}, {2}, {3}, {4}, {∞}, 01021 = 0(1324∞)4104 = (1324∞)04104 = (1∞304)402∞0 = (10∞)(234)41∞ and $(41\infty)^{\pi}$ = 012 where π = (2∞)(04) ∈ N 60

[01021]=[012]

01023 = (0∞4)(132)3412 and $(3412)^{\pi}$ = 0134 where π = (13024) ∈ N

[01023]=[0134]

01024 = (021)(4∞3)3131 and $(3131)^{\pi}$ = 0101 where π = (03)(4∞) ∈ N

[01024]=[0101]

0102∞ = 0(0∞124)014 = (0∞124)∞014 =
(204∞30)∞240 and (∞240)$^{\pi}$ = 0124 where
π = (1∞042) $\in$ N
[0102∞]=[0124]

[0103] $N^{(0\ 1\ 0\ 3)}$ = <*e*> has orbits {0}, {1}, {2}, {3}, 60
{4}, {∞}, 01031 = 0(310)(24∞)130 =
(310)(24∞)3130 and (3130)$^{\pi}$ = 0103 where
π = (03)(4∞) $\in$ N
[01031]=[0103]

01032 = (03)(12)0103
[01032]=[0103]

0103∞ = (01∞)(432)4∞0 and (4∞0)$^{\pi}$ = 013
where π = (1403∞) $\in$ N
[0103∞]=[013]

01030 = (01)(24)0304 and (0304)$^{\pi}$ = 0102
where π = (1∞423) $\in$ N
[01030]=[0124]

01034 = (014)(∞34)0∞43 and (0∞430$^{\pi}$ = 0134
where π = (1∞)(34) $\in$ N
[01034]=[0134]

[0101] $N^{(0\ 1\ 0\ 1)}$ = <(24)(3∞)> has orbits {0}, {1}, 30
{2, 4}, {3, ∞}, 01012 = (42)(∞3)41∞0 and
(41∞0)$^{\pi}$ = 0124 where π = (2∞)(04) $\in$ N
[01012]=[0124]

01013 = (023)(14∞)∞1∞4 and (∞1∞4)$^{\pi}$ = 0102
where π = (2∞034) $\in$ N
[01013]=[0102]

[01010] $N^{(0\ 1\ 0\ 1\ 0)}$ = <(24)(3∞), (143)(2∞0), 5
(14∞)(230)> has orbit {0, 1, 2, 3, 4, ∞}
01010 ~ 10101 ~ ∞3∞3∞ ~ 3∞3∞3 ~ 24242 ~
42424 and for all m in {0, 1, 2, 3, 4, ∞}
[01010m]=[0101]

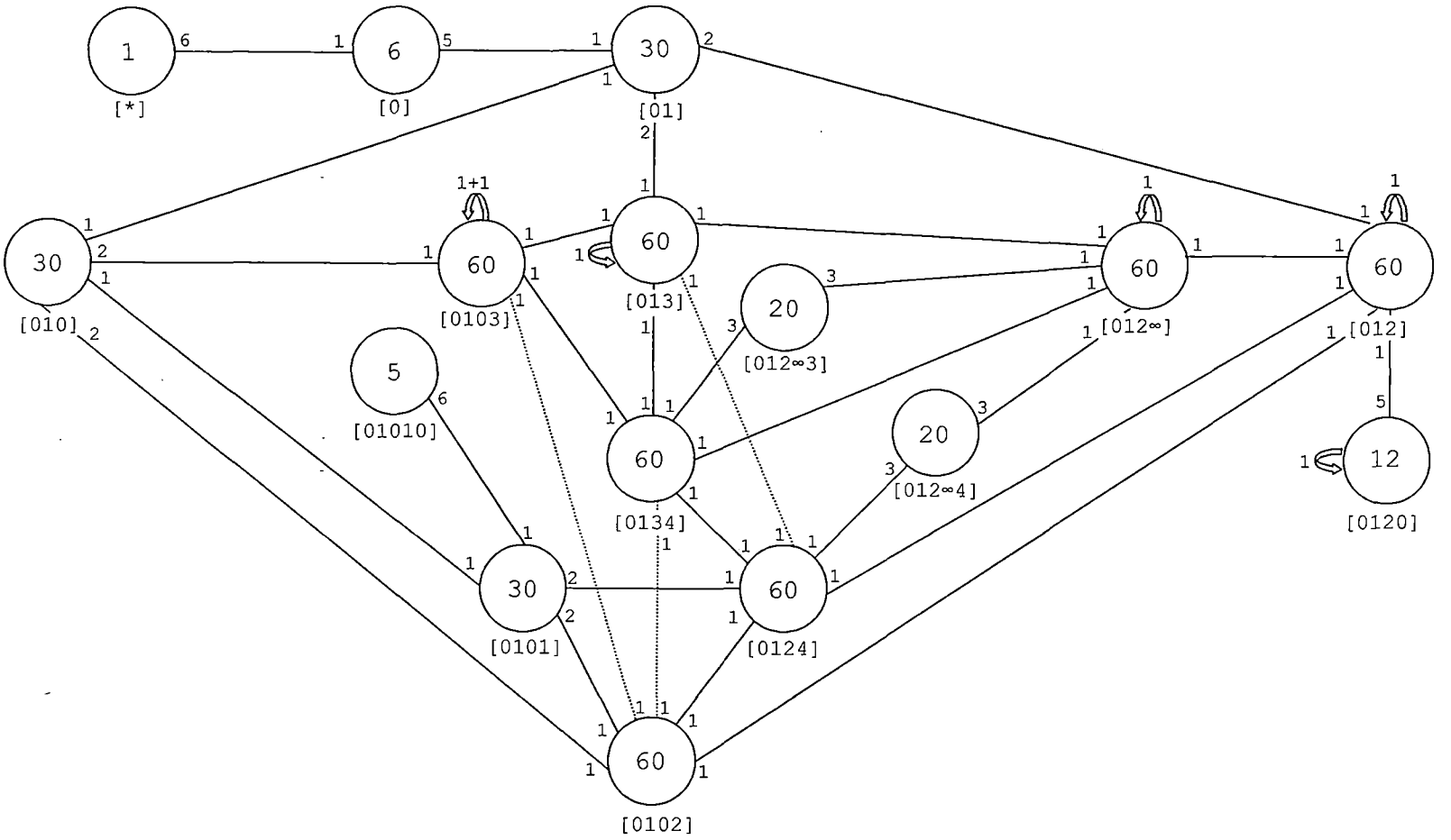


Figure 7.1 Cayley Graph of $L_2(41)$ over $L_2(5)$

The Cayley graph of $L_2(41)$ over $L_2(5)$ has as vertices the set of right cosets of N in G, i.e. the set Nw_i, where w_i are words in T. The diagram above illustrates each orbit of N in its action on the vertices by right multiplication which is represented by a single node, labeled with the number of vertices that it contains. Lines in between these nodes are labeled with integers to denote how many edges from a vertex of one node will lead to the vertices of the other.

At this point we can calculate the action of the six symmetric generators on the cosets of $L_2(41)$ over $L_2(5)$ (see Appendix B) for the labeling used).

The action of the symmetric generators on the cosets of N in G is given by:

t_0 = (1, 2)(3, 7)(4, 9)(5, 10)(6, 11)(8, 16)(12, 28)(13, 30)
(14, 32)(15, 34)(17, 37)(18, 39)(19, 41)(20, 42)(21,
44)(22, 46)(23, 48)(24, 49)(25, 50)(26, 51)(27, 52)
(29, 57)(31, 62)(33, 67)(35, 71)(36, 72)(38, 76)(40,
81)(43, 89)(45, 94)(47, 97)(53, 84)(54, 117)(55, 119)
(56, 121)(58, 127)(59, 96)(60, 130)(61, 132)(63, 135)
(64, 137)(65, 139)(66, 141)(68, 144)(69, 146)(70, 148)
(73, 156)(74, 158)(75, 160)(77, 166)(78, 168)(79, 170)
(80, 172)(82, 175)(83, 177)(85, 180)(86, 182)(87, 184)
(88, 186)(90, 190)(91, 129)(92, 193)(93, 195)(95, 126)
(98, 202)(99, 204)(100, 205)(101, 207)(102, 209)(103,
210)(104, 211)(105, 213)(106, 215)(107, 217)(108, 219)
(109, 220)(110, 221)(111, 223)(112, 225)(113, 227)
(114, 228)(115, 229)(116, 230)(118, 235)(120, 241)
(122, 246)(123, 214)(124, 206)(125, 247)(128, 252)
(131, 258)(133, 216)(134, 262)(136, 267)(138, 272)
(140, 277)(142, 281)(143, 244)(145, 286)(147, 292)

(149, 294)(150, 295)(151, 181)(152, 296)(153, 297)
(154, 257)(155, 298)(157, 256)(159, 243)(161, 309)
(162, 310)(163, 311)(164, 192)(165, 312)(167, 317)
(169, 322)(171, 326)(173, 329)(174, 330)(176, 333)
(178, 338)(179, 339)(183, 343)(185, 346)(187, 349)
(188, 350)(189, 351)(191, 354)(194, 358)(196, 361)
(197, 362)(198, 363)(199, 342)(200, 238)(201, 364)
(203, 366)(208, 373)(212, 377)(218, 268)(222, 231)
(224, 396)(226, 401)(232, 348)(233, 409)(234, 411)
(236, 413)(237, 415)(239, 417)(240, 419)(242, 287)
(245, 422)(248, 356)(249, 429)(250, 430)(251, 431)
(253, 433)(254, 435)(255, 321)(259, 436)(260, 441)
(261, 320)(263, 389)(264, 444)(265, 446)(266, 448)
(270, 273)(271, 452)(274, 383)(275, 313)(276, 455)
(278, 457)(279, 458)(280, 459)(282, 462) (283, 293)
(284, 369)(285, 370)(288, 328)(289, 385)(290, 386)
(291, 467)(299, 471)(300, 424)(301, 475)(302, 352)
(303, 479)(304, 331)(305, 316)(306, 482)(307, 477)
(308, 468)(314, 491)(315, 478)(318, 493)(319, 382)
(323, 466)(327, 499)(332, 502)(334, 505)(335, 506)
(336, 423)(337, 507)(340, 510)(341, 403)(344, 514)
(345, 408)(347, 407)(353, 416)(355, 518)(357, 519)
(359, 520)(360, 521)(365, 376)(367, 432)(368, 524)
(371, 511)(372, 404)(374, 527)(375, 525)(378, 495)
(379, 529)(380, 530)(381, 492)(384, 390)(387, 398)
(388, 481)(391, 534)(392, 535)(393, 536)(395, 515)
(397, 449)(399, 540)(400, 541)(402, 509)(405, 542)
(406, 543)(410, 545)(412, 546)(414, 501)(418, 494)
(420, 480)(421, 500)(425, 549)(426, 550)(427, 551)
(428, 552)(437, 537)(439, 556)(440, 557)(442, 456)
(443, 503)(445, 453)(447, 469)(451, 562)(454, 496)
(460, 563)(461, 564)(464, 522)(470, 566)(472, 547)
(473, 567)(474, 568)(483, 561)(484, 523)(485, 488)
(486, 489)(487, 497)(490, 548)(498, 572)(504, 571)
(508, 555)(512, 538)(513, 573)(516, 544)(517, 554)
(526, 539)(528, 531)(532, 570)(533, 569)(553, 565)
(558, 574)(559, 560)

t_1 = (1, 3)(2, 17)(4, 23)(5, 13)(6, 12)(7, 63)(8, 18)(9, 82)
(10, 90)(11, 101)(14, 122)(15, 59)(16, 111)(19, 161)
(20, 78)(21, 151)(22, 68)(24, 187)(25, 86)(26, 53)(27,
54)(28, 237)(29, 58)(30, 254)(31, 64)(32, 273)(33,
107) (34, 287)(35, 108)(36, 73)(37, 303)(38, 77)(39,
284)(40, 83)(41, 114)(42, 344)(43, 91)(44, 336)(45,

102)(46, 61)(47, 98)(48, 367)(49, 297)(50, 310)(51,
390)(52, 397)(55, 241)(56, 232)(57, 105)(60, 209)(62,
198)(65, 449)(66, 269)(67, 250)(69, 332)(70, 264)(71,
469)(72, 346)(74, 121)(75, 283)(76, 482)(79, 184)(80,
319)(81, 462)(84, 473)(85, 335)(87, 376)(88, 262)(89,
475)(92, 517)(93, 128)(94, 351)(95, 511)(96, 327)(97,
416)(99, 153)(100, 341)(103, 220)(104, 375)(106, 242)
(109, 450)(110, 387)(112, 407)(113, 259)(115, 178)
(116, 231)(117, 497)(118, 238)(119, 286)(120, 404)
(123, 512)(124, 202)(125, 248)(126, 249)(127, 358)
(129, 255)(130, 188)(131, 423)(132, 495)(133, 424)
(134, 263)(135, 479)(136, 268)(137, 377)(138, 274)
(139, 490)(140, 278)(141, 354)(142, 315)(143, 282)
(144, 205)(145, 288)(146, 217)(147, 240)(148, 285)
(149, 171)(150, 384)(152, 444)(154, 299)(155, 300)
(156, 516)(157, 304)(158, 381)(159, 438)(160, 422)
(162, 350)(163, 359)(164, 313)(165, 314)(166, 457)
(167, 318)(168, 257)(169, 323)(172, 290)(173, 191)
(174, 331)(175, 388)(176, 334)(177, 548)(179, 330)
(180, 460)(181, 340)(182, 218)(183, 345)(185, 236)
(186, 234)(189, 267)(190, 239)(192, 355)(193, 427)
(194, 470)(195, 572)(196, 412)(197, 296)(199, 391)
(200, 464)(201, 365)(203, 301)(204, 544)(206, 279)
(207, 523)(208, 374)(210, 230)(211, 245)(212, 378)
(213, 491)(214, 246)(215, 321)(216, 383)(219, 419)
(221, 536)(222, 394)(223, 420)(224, 398)(225, 543)
(226, 382)(227, 342)(228, 566)(229, 553)(233, 534)
(235, 485)(243, 329)(244, 487)(247, 521)(251, 554)
(252, 456)(253, 364)(256, 395)(258, 567)(261, 349)
(265, 333)(266, 418)(270, 458)(275, 443)(276, 431)
(277, 447)(280, 434)(281, 530)(291, 343)(292, 306)
(293, 372)(294, 477)(295, 510)(302, 426)(305, 371)
(307, 405)(308, 560)(309, 312)(311, 322)(316, 564)
(317, 392)(320, 413)(324, 565)(325, 465)(326, 571)
(328, 525)(337, 493)(338, 549)(339, 370)(347, 529)
(348, 399)(353, 538)(356, 557)(357, 574)(360, 501)
(361, 559)(362, 515)(363, 403)(366, 499)(368, 463)
(369, 417)(373, 556)(379, 414)(380, 494)(385, 506)
(386, 393)(389, 400)(396, 451)(401, 573)(402, 468)
(406, 496)(408, 532)(409, 466)(410, 429)(411, 569)
(415, 481)(421, 483)(425, 476)(428, 508)(430, 471)
(432, 484)(435, 480)(436, 489)(439, 500)(440, 454)
(441, 563)(442, 474)(445, 446)(448, 527)(452, 558)
(453, 531)(455, 535)(459, 555)(461, 518)(467, 526)
(472, 551)(478, 561)(486, 520)(488, 541)(492, 570)

(498, 542)(502, 514)(503, 545)(504, 568)(505, 509)
(507, 547)(513, 562)(519, 550)(522, 533)(524, 552)
(528, 546)(539, 540)

t_2 = (1, 6)(2, 47)(3, 36)(4, 38)(5, 29)(7, 245)(8, 26)(9,
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116)(16, 218)(17, 174)(18, 206)(19, 176)(20, 143)(21,
208)(22, 128)(23, 222)(24, 191)(25, 167)(27, 181)(28,
241)(30, 374)(31, 201)(32, 353)(33, 114)(34, 370)(35,
136)(37, 242)(39, 344)(40, 154)(41, 295)(42, 210)(43,
164)(44, 441)(45, 125)(46, 523)(48, 197)(49, 390)(50,
102)(51, 271)(52, 234)(53, 414)(54, 482)(55, 487)(56,
103)(57, 547)(58, 434)(59, 538)(60, 548)(61, 229)(62,
544)(63, 377)(64, 525)(65, 453)(66, 443)(67, 166)(68,
223)(69, 418)(70, 172)(71, 79)(72, 275)(73, 478)(74,
537)(75, 204)(76, 475)(77, 144)(78, 113)(80, 228)(81,
224)(82, 120)(83, 376)(84, 330)(85, 274)(86, 513)(87,
118)(88, 389)(89, 254)(90, 177)(91, 415)(92, 528)(93,
476)(94, 486)(95, 504)(96, 494)(97, 559)(98, 456)(99,
519)(100, 346)(101, 296)(104, 420)(105, 211)(106, 392)
(107, 408)(108, 109)(110, 532)(111, 333)(112, 380)
(115, 365)(117, 568)(119, 267)(121, 462)(122, 127)
(123, 428)(124, 440)(126, 502)(129, 460)(130, 261)
(131, 427)(133, 451)(135, 378)(137, 354)(138, 442)
(139, 484)(140, 260)(141, 178)(142, 156)(145, 323)
(147, 324)(148, 150)(149, 410)(151, 470)(152, 471)
(153, 195)(155, 371)(157, 512)(158, 207)(159, 477)
(160, 509)(161, 565)(162, 514)(163, 226)(165, 347)
(168, 238)(169, 340)(170, 539)(171, 490)(173, 436)
(175, 386)(179, 435)(180, 230)(182, 266)(183, 526)
(184, 493)(185, 455)(186, 491)(187, 516)(188, 480)
(192, 289)(193, 564)(194, 288)(196, 416)(198, 531)
(199, 411)(200, 465)(202, 203)(205, 286)(209, 510)
(212, 356)(213, 302)(214, 355)(215, 303)(217, 572)
(219, 331)(220, 521)(221, 391)(225, 317)(227, 292)
(231, 322)(232, 540)(233, 553)(235, 541)(237, 464)
(239, 492)(240, 311)(243, 527)(244, 290)(246, 297)
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(263, 561)(264, 301)(265, 300)(268, 444)(269, 463)
(270, 350)(272, 352)(273, 366)(276, 437)(277, 382)
(278, 398)(279, 405)(280, 483)(281, 336)(285, 501)
(287, 383)(294, 358)(299, 466)(304, 439)(305, 313)
(306, 446)(308, 452)(309, 412)(310, 566)(312, 555)
(314, 424)(315, 425)(316, 413)(318, 438)(319, 364)

(320, 461)(321, 552)(325, 454)(326, 505)(327, 363)
(328, 507)(332, 481)(334, 499)(335, 406)(337, 551)
(339, 562)(341, 349)(342, 557)(345, 567)(348, 569)
(351, 445)(357, 426)(359, 571)(360, 529)(361, 385)
(362, 530)(367, 409)(368, 473)(369, 533)(372, 488)
(373, 403)(375, 489)(379, 419)(381, 573)(384, 511)
(387, 535)(388, 536)(393, 563)(394, 421)(396, 497)
(397, 518)(399, 515)(400, 450)(401, 429)(402, 554)
(404, 546)(407, 542)(417, 479)(422, 469)(423, 574)
(431, 549)(432, 534)(433, 468)(447, 472)(448, 457)
(458, 498)(459, 474)(467, 560)(485, 570)(495, 550)
(496, 506)(500, 543)(503, 558)(508, 524)(520, 522)

t_3 = (1, 8)(2, 43)(3, 31)(4, 33)(5, 45)(6, 40)(7, 198)(9,
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185)(16, 145)(17, 129)(18, 157)(19, 159)(20, 226)(21,
131)(22, 214)(23, 169)(24, 171)(25, 147)(26, 138)(27,
199)(28, 311)(29, 194)(30, 499)(32, 369)(34, 452)(35,
180)(36, 179)(37, 395)(38, 178)(39, 111)(41, 330)(42,
196)(44, 522)(46, 337)(47, 192)(48, 485)(49, 106)(50,
202)(51, 218)(52, 503)(53, 241)(54, 208)(55, 174)(56,
230)(57, 251)(58, 526)(59, 406)(60, 281)(61, 339)(62,
270)(63, 328)(64, 445)(65, 447)(66, 327)(67, 299)(68,
368)(69, 170)(70, 367)(71, 233)(72, 294)(73, 420)(74,
422)(75, 207)(76, 562)(77, 513)(78, 405)(79, 366)(80,
186)(81, 285)(82, 387)(83, 139)(84, 280)(85, 474)(86,
297)(87, 388)(88, 376)(89, 483)(90, 495)(91, 516)(92,
465)(93, 558)(94, 525)(95, 496)(96, 404)(97, 352)(98,
100)(99, 246)(101, 408)(102, 515)(103, 278)(104, 221)
(105, 410)(107, 293)(108, 409)(109, 533)(110, 269)
(112, 165)(113, 232)(114, 302)(115, 316)(116, 415)
(117, 540)(119, 321)(121, 467)(122, 313)(123, 439)
(124, 421)(125, 437)(126, 524)(127, 534)(128, 507)
(132, 155)(133, 455)(134, 444)(135, 204)(136, 454)
(137, 475)(140, 484)(141, 509)(142, 365)(143, 394)
(146, 149)(148, 358)(150, 167)(151, 360)(152, 193)
(153, 453)(154, 457)(156, 443)(158, 490)(160, 382)
(161, 162)(163, 298)(164, 282)(166, 247)(168, 401)
(173, 519)(175, 256)(176, 210)(177, 250)(181, 498)
(182, 435)(184, 557)(187, 434)(188, 547)(189, 334)
(190, 449)(191, 379)(195, 461)(197, 359)(200, 451)
(201, 277)(205, 385)(206, 346)(209, 462)(211, 296)
(213, 266)(215, 542)(216, 508)(217, 351)(219, 556)
(220, 393)(222, 378)(223, 531)(225, 349)(227, 563)
(228, 544)(229, 567)(231, 551)(234, 505)(235, 492)

(236, 528)(237, 288)(238, 381)(239, 310)(240, 363)
(242, 565)(243, 289)(244, 312)(245, 248)(249, 504)
(252, 568)(253, 539)(254, 257)(255, 292)(258, 480)
(260, 433)(262, 464)(263, 471)(264, 342)(265, 362)
(268, 398)(271, 459)(272, 413)(273, 307)(274, 324)
(275, 323)(276, 527)(279, 303)(283, 423)(284, 500)
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(314, 418)(315, 377)(317, 506)(318, 416)(319, 549)
(320, 572)(322, 350)(325, 399)(326, 354)(329, 518)
(331, 403)(332, 384)(335, 566)(336, 573)(338, 402)
(341, 545)(343, 389)(344, 569)(345, 400)(347, 493)
(348, 530)(355, 424)(356, 425)(357, 491)(361, 529)
(364, 486)(370, 419)(371, 514)(373, 550)(374, 535)
(375, 478)(383, 571)(386, 521)(390, 574)(391, 397)
(392, 543)(396, 477)(407, 553)(411, 537)(412, 555)
(414, 559)(417, 470)(426, 463)(427, 441)(428, 510)
(429, 538)(430, 570)(431, 472)(432, 488)(438, 512)
(440, 501)(442, 448)(450, 476)(456, 497)(458, 473)
(460, 494)(469, 517)(479, 481)(482, 546)(487, 520)
(489, 554)(502, 536)(511, 564)(523, 532)(541, 548)

t_4 = (1, 4)(2, 14)(3, 24)(5, 21)(6, 35)(7, 55)(8, 19)(9, 74)
(10, 60)(11, 79)(12, 109)(13, 123)(15, 95)(16, 99)(17,
152)(18, 162)(20, 105)(22, 87)(23, 200)(25, 112)(26,
142)(27, 69)(28, 179)(29, 196)(30, 250)(31, 149)(32,
265)(33, 65)(34, 284)(36, 173)(37, 301)(38, 188)(39,
315)(40, 84)(41, 276)(42, 219)(43, 103)(44, 352)(45,
92)(46, 320)(47, 133)(48, 211)(49, 371)(50, 233)(51,
385)(52, 80)(53, 314)(54, 365)(56, 381)(57, 258)(58,
425)(59, 432)(61, 402)(62, 388)(63, 205)(64, 190)(66,
456)(67, 230)(68, 372)(70, 279)(71, 287)(72, 336)(73,
261)(75, 259)(76, 253)(77, 222)(78, 348)(81, 485)(82,
330)(83, 469)(85, 210)(86, 94)(88, 201)(89, 515)(90,
361)(91, 458)(93, 487)(96, 335)(97, 270)(98, 309)(100,
347)(101, 104)(102, 393)(106, 246)(107, 169)(108, 137)
(110, 199)(111, 117)(113, 473)(114, 241)(115, 243)
(116, 183)(118, 522)(119, 482)(120, 239)(121, 146)
(122, 477)(124, 249)(125, 439)(126, 260)(127, 334)
(128, 461)(129, 500)(130, 331)(131, 256)(132, 255)
(134, 325)(135, 341)(136, 417)(138, 275)(139, 451)
(140, 476)(141, 478)(143, 468)(144, 329)(145, 413)
(147, 289)(150, 392)(151, 454)(153, 472)(154, 421)
(155, 307)(156, 378)(157, 396)(158, 446)(159, 305)
(160, 377)(161, 463)(163, 488)(164, 498)(165, 283)

(166, 444)(167, 509)(168, 571)(170, 453)(171, 324)
(172, 548)(174, 502)(175, 467)(176, 273)(177, 542)
(178, 376)(180, 300)(181, 296)(182, 567)(184, 398)
(185, 326)(186, 569)(187, 527)(189, 430)(191, 466)
(192, 471)(193, 247)(194, 356)(195, 546)(197, 369)
(198, 338)(202, 346)(203, 427)(204, 503)(206, 508)
(207, 562)(208, 349)(209, 483)(212, 297)(213, 224)
(214, 379)(215, 416)(216, 290)(217, 262)(218, 497)
(220, 227)(221, 518)(225, 272)(226, 399)(228, 452)
(229, 337)(231, 543)(232, 510)(234, 540)(235, 566)
(236, 357)(237, 489)(238, 449)(240, 319)(242, 304)
(244, 481)(245, 490)(248, 568)(251, 535)(252, 312)
(263, 367)(264, 380)(266, 368)(267, 353)(268, 516)
(269, 517)(271, 298)(274, 559)(277, 322)(278, 342)
(280, 340)(281, 495)(286, 528)(291, 437)(292, 387)
(293, 431)(294, 442)(295, 354)(299, 484)(302, 445)
(303, 366)(306, 455)(308, 310)(311, 529)(313, 423)
(316, 426)(317, 545)(318, 407)(321, 412)(323, 514)
(332, 436)(333, 441)(339, 556)(343, 397)(344, 564)
(345, 570)(350, 422)(351, 491)(355, 530)(358, 552)
(359, 401)(360, 460)(362, 386)(364, 554)(370, 532)
(373, 390)(374, 565)(375, 505)(382, 551)(383, 470)
(384, 550)(389, 534)(391, 572)(394, 533)(395, 496)
(400, 435)(403, 574)(404, 409)(405, 524)(406, 557)
(408, 541)(410, 544)(411, 525)(414, 465)(415, 520)
(418, 450)(419, 561)(420, 539)(424, 475)(428, 459)
(429, 462)(433, 492)(434, 438)(440, 457)(443, 511)
(447, 560)(448, 464)(474, 512)(479, 480)(486, 549)
(493, 531)(494, 521)(499, 555)(501, 563)(504, 538)
(506, 513)(507, 553)(519, 558)(523, 573)(526, 536)
(537, 547)

t_∞ = (1, 5)(2, 15)(3, 22)(4, 27)(6, 20)(7, 61)(8, 25)(9, 80)
(10, 75)(11, 100)(12, 113)(13, 70)(14, 126)(16, 56)
(17, 96)(18, 88)(19, 165)(21, 155)(23, 106)(24, 197)
(26, 150)(28, 234)(29, 66)(30, 90)(31, 124)(32, 271)
(33, 110)(34, 266)(35, 115)(36, 93)(37, 285)(38, 85)
(39, 321)(40, 163)(41, 337)(42, 332)(43, 153)(44, 60)
(45, 140)(46, 353)(47, 104)(48, 342)(49, 252)(50, 372)
(51, 389)(52, 395)(53, 406)(54, 244)(55, 407)(57, 296)
(58, 189)(59, 261)(62, 251)(63, 293)(64, 280)(65, 450)
(67, 176)(68, 308)(69, 419)(71, 386)(72, 203)(73,
463)(74, 436)(76, 486)(77, 489)(78, 328)(79, 492)(81,
316)(82, 348)(83, 142)(84, 504)(86, 370)(87, 292)(89,
302)(91, 360)(92, 131)(94, 246)(95, 449)(97, 435)(98,

511)(99, 509)(101, 382)(102, 346)(103, 432)(105, 452)
(107, 474)(108, 393)(109, 341)(111, 403)(112, 537)
(114, 461)(116, 240)(117, 344)(118, 365)(119, 291)
(120, 401)(121, 522)(122, 212)(123, 297)(125, 194)
(127, 555)(128, 257)(129, 375)(130, 319)(132, 278)
(133, 428)(134, 290)(135, 413)(136, 276)(138, 363)
(139, 358)(141, 260)(143, 306)(144, 287)(145, 175)
(146, 284)(147, 481)(148, 207)(149, 431)(151, 211)
(152, 245)(154, 433)(156, 247)(157, 237)(158, 343)
(159, 255)(160, 448)(161, 381)(162, 412)(164, 501)
(166, 530)(167, 325)(168, 310)(169, 345)(170, 289)
(171, 426)(172, 456)(173, 440)(174, 317)(177, 570)
(178, 503)(179, 391)(180, 405)(181, 368)(182, 397)
(183, 210)(184, 233)(185, 468)(186, 539)(187, 239)
(188, 414)(190, 402)(191, 357)(192, 459)(193, 563)
(195, 329)(196, 230)(198, 362)(199, 272)(200, 331)
(201, 523)(202, 236)(204, 298)(205, 374)(206, 380)
(208, 376)(209, 369)(213, 320)(214, 418)(215, 253)
(216, 472)(217, 221)(218, 373)(219, 225)(220, 535)
(222, 400)(223, 282)(224, 235)(226, 438)(227, 267)
(228, 483)(229, 531)(231, 513)(232, 533)(238, 263)
(241, 330)(242, 361)(243, 528)(248, 549)(249, 383)
(250, 541)(254, 494)(256, 312)(258, 453)(259, 371)
(262, 526)(264, 560)(265, 454)(269, 279)(273, 517)
(274, 439)(275, 430)(277, 327)(281, 574)(283, 367)
(286, 387)(288, 384)(294, 542)(295, 417)(299, 423)
(300, 476)(301, 425)(303, 512)(304, 311)(305, 349)
(307, 561)(309, 520)(313, 545)(314, 480)(315, 551)
(318, 482)(322, 506)(324, 394)(326, 410)(333, 557)
(334, 552)(335, 500)(336, 550)(338, 510)(339, 559)
(340, 538)(347, 565)(351, 420)(352, 524)(354, 429)
(355, 498)(356, 554)(359, 462)(364, 529)(366, 568)
(377, 441)(378, 564)(379, 496)(385, 392)(388, 399)
(390, 536)(398, 553)(404, 527)(408, 556)(409, 493)
(411, 415)(416, 442)(421, 470)(422, 499)(424, 572)
(427, 521)(434, 558)(437, 488)(443, 446)(444, 495)
(445, 473)(447, 525)(455, 507)(457, 515)(458, 518)
(460, 491)(464, 502)(465, 532)(467, 487)(469, 505)
(471, 573)(477, 548)(478, 547)(479, 484)(485, 546)
(490, 534)(497, 569)(508, 567)(514, 540)(519, 566)
(543, 562)(544, 571).

Representation of Elements of $L_2(41)$

Every element of $L_2(41)$ can be represented as πw, where π is a permutation of $L_2(5)$, on 6 letters, and w is a product of at most five symmetric generators.

For example:

Let α = (1, 25, 481, 484, 45)(2, 153, 258, 314, 214, 3, 124, 470, 295, 185)(4, 110, 279, 512, 226, 6, 163, 204, 413, 199)(5, 140, 479, 147, 8)(7, 362, 454, 276, 404, 17, 375, 547, 414, 339)(9, 122, 545, 109, 232, 12, 72, 542, 253, 186)(10, 210, 67, 423, 367, 13, 365, 83, 358, 207)(11, 235, 79, 568, 49, 23, 345, 222, 564, 98)(14, 401, 310, 187, 558, 36, 391, 182, 97, 524)(15, 468, 417, 421, 31, 22, 418, 480, 453, 43)(16, 175, 312, 54, 376, 18, 237, 384, 42, 230)(19, 255, 87, 399, 167, 26, 363, 116, 411, 112)(20, 438, 303, 269, 33, 27, 272, 135, 298, 40)(21, 92, 532, 201, 327, 29, 125, 488, 103, 132)(24, 426, 73, 351, 221, 47, 459, 32, 209, 359)(28, 304, 443, 247, 530, 82, 286, 307, 517, 505)(30, 422, 436, 74, 499, 90, 444, 290, 134, 495)(34, 105, 326, 429, 340, 68, 181, 355, 572, 213)(35, 405, 328, 293, 474, 38, 503, 395, 285, 316)(37, 52, 178, 85, 107, 63, 78, 180, 115, 81)(39, 403, 200, 451, 331, 111, 321, 291, 466, 119)(41, 241, 406, 261, 59, 53, 330, 337, 353, 46)(44, 121, 487, 309, 334, 58, 262, 502, 390, 281)(48, 546, 318, 442, 160, 101, 556, 225, 305, 264)(50, 236, 243, 170, 419, 86, 123, 274, 394, 306)(51, 373, 336, 471, 238, 161, 412, 127, 490, 343)(55, 317, 322, 350, 506, 174, 407, 398, 268, 553)(56, 196, 332, 288, 157, 88, 208, 244, 256, 145)(57, 62, 270, 251, 296, 151, 91, 516, 360, 211)(60, 574, 536, 464, 526, 189, 552, 520, 467, 522)(61, 559, 188, 478, 129, 96, 527, 136, 265, 198)(64, 473, 518, 195, 114, 89, 228, 571, 249, 84)(65, 525, 246, 509, 260, 154, 515, 346, 380, 476)(66, 277, 523, 465, 131, 155, 278, 432, 437, 194)(69, 289, 528, 202, 372, 143, 324, 439, 297, 370)(70, 283, 299, 176, 183, 75, 148, 139, 142, 118)(71, 184, 333, 425, 554, 77, 231, 315, 441, 521)(76, 543, 385,

374, 220, 108, 493, 565, 361, 364)(80, 539, 215,
294, 203, 113, 533, 341, 313, 212)(93, 434, 239,
168, 120, 126, 352, 435, 397, 179)(94, 447, 450,
300, 206, 102, 457, 433, 141, 99)(95, 379, 357,
460, 254, 128, 455, 428, 338, 190)(100, 511, 378,
400, 169, 106, 252, 366, 492, 224)(104, 217, 420,
463, 171, 197, 462, 369, 271, 192)(117, 514, 259,
371, 540, 344, 497, 172, 456, 569)(130, 319, 248,
152, 563, 267, 227, 193, 245, 549)(133, 507, 257,
494, 491, 191, 496, 449, 402, 510)(137, 475)(138,
150, 325, 388, 292, 159, 165, 537, 415, 240)(144,
287)(146, 431, 216, 567, 531, 282, 501, 173, 566,
500)(149, 284, 335, 519, 440, 164, 223, 229, 508,
472)(156, 446, 311, 234, 469, 273, 561, 387, 348,
166)(158, 534, 555, 162, 381, 263, 573, 550, 218,
389)(177, 541, 477, 396, 548, 250, 570, 275, 323,
430)(205, 392, 562, 486, 529, 242, 347, 409, 393,
535)(219, 408, 382, 448, 416, 482, 485, 342, 560,
349)(233, 386, 427, 377, 551, 513, 489, 356, 301,
557)(266, 320, 424, 498, 368, 308, 538, 354, 410,
452)(280, 504, 383, 544, 483, 302, 461, 329, 458,
445),

for $\alpha \in L_2(41)$, where α is a permutation on 574 letters.

Now $N^{\alpha} = N\alpha$

$\Rightarrow$ $1^{\alpha} = 25$ (Since N is labeled as 1 and 1 goes to 25 under α)

$\Rightarrow$ $N\alpha = Nt_3t_\infty$ (Since Nt_3t_∞ is labeled as 25)

$\Rightarrow$ $N\alpha t_\infty t_3 = N$

$\Rightarrow$ $\alpha t_3 t_\infty = n$, for $n \in L_2(5)$

$\Rightarrow$ $\alpha = nt_3t_\infty$, a symmetric representation for α.

Now, what is n?

$\alpha t_\infty t_3$ = (1, 25, 481, 484, 45)(2, 153, 258, 314, 214, 3, 124,
470, 295, 185)(4, 110, 279, 512, 226, 6, 163, 204,
413, 199)(5, 140, 479, 147, 8)(7, 362, 454, 276,
404, 17, 375, 547, 414, 339)(9, 122, 545, 109, 232,

12, 72, 542, 253, 186)(10, 210, 67, 423, 367, 13,
365, 83, 358, 207)(11, 235, 79, 568, 49, 23, 345,
222, 564, 98)(14, 401, 310, 187, 558, 36, 391, 182,
97, 524)(15, 468, 417, 421, 31, 22, 418, 480, 453,
43)(16, 175, 312, 54, 376, 18, 237, 384, 42, 230)
(19, 255, 87, 399, 167, 26, 363, 116, 411, 112)(20,
438, 303, 269, 33, 27, 272, 135, 298, 40)(21, 92,
532, 201, 327, 29, 125, 488, 103, 132)(24, 426, 73,
351, 221, 47, 459, 32, 209, 359)(28, 304, 443, 247,
530, 82, 286, 307, 517, 505)(30, 422, 436, 74, 499,
90, 444, 290, 134, 495)(34, 105, 326, 429, 340, 68,
181, 355, 572, 213)(35, 405, 328, 293, 474, 38,
503, 395, 285, 316)(37, 52, 178, 85, 107, 63, 78,
180, 115, 81)(39, 403, 200, 451, 331, 111, 321,
291, 466, 119)(41, 241, 406, 261, 59, 53, 330, 337,
353, 46)(44, 121, 487, 309, 334, 58, 262, 502, 390,
281)(48, 546, 318, 442, 160, 101, 556, 225, 305,
264)(50, 236, 243, 170, 419, 86, 123, 274, 394,
306)(51, 373, 336, 471, 238, 161, 412, 127, 490,
343)(55, 317, 322, 350, 506, 174, 407, 398, 268,
553)(56, 196, 332, 288, 157, 88, 208, 244, 256,
145)(57, 62, 270, 251, 296, 151, 91, 516, 360, 211)
(60, 574, 536, 464, 526, 189, 552, 520, 467, 522)
(61, 559, 188, 478, 129, 96, 527, 136, 265, 198)
(64, 473, 518, 195, 114, 89, 228, 571, 249, 84)(65,
525, 246, 509, 260, 154, 515, 346, 380, 476)(66,
277, 523, 465, 131, 155, 278, 432, 437, 194)(69,
289, 528, 202, 372, 143, 324, 439, 297, 370)(70,
283, 299, 176, 183, 75, 148, 139, 142, 118)(71,
184, 333, 425, 554, 77, 231, 315, 441, 521)(76,
543, 385, 374, 220, 108, 493, 565, 361, 364)(80,
539, 215, 294, 203, 113, 533, 341, 313, 212)(93,
434, 239, 168, 120, 126, 352, 435, 397, 179)(94,
447, 450, 300, 206, 102, 457, 433, 141, 99)(95,
379, 357, 460, 254, 128, 455, 428, 338, 190)(100,
511, 378, 400, 169, 106, 252, 366, 492, 224)(104,
217, 420, 463, 171, 197, 462, 369, 271, 192)(117,
514, 259, 371, 540, 344, 497, 172, 456, 569)(130,
319, 248, 152, 563, 267, 227, 193, 245, 549)(133,
507, 257, 494, 491, 191, 496, 449, 402, 510)(137,
475)(138, 150, 325, 388, 292, 159, 165, 537, 415,
240)(144, 287)(146, 431, 216, 567, 531, 282, 501,
173, 566, 500)(149, 284, 335, 519, 440, 164, 223,
229, 508, 472)(156, 446, 311, 234, 469, 273, 561,
387, 348, 166)(158, 534, 555, 162, 381, 263, 573,

550, 218, 389)(177, 541, 477, 396, 548, 250, 570,
275, 323, 430)(205, 392, 562, 486, 529, 242, 347,
409, 393, 535)(219, 408, 382, 448, 416, 482, 485,
342, 560, 349)(233, 386, 427, 377, 551, 513, 489,
356, 301, 557)(266, 320, 424, 498, 368, 308, 538,
354, 410, 452)(280, 504, 383, 544, 483, 302, 461,
329, 458, 445)$t_\infty t_3$

$= (Id)$

Therefore, the action of $\alpha t_\infty t_3$ on the symmetric generators is given by:

$Nt_0 = 2 \rightarrow 2 = Nt_0$

$Nt_1 = 3 \rightarrow 3 = Nt_1$

$Nt_2 = 6 \rightarrow 6 = Nt_2$

$Nt_3 = 8 \rightarrow 8 = Nt_3$

$Nt_4 = 4 \rightarrow 4 = Nt_4$

$Nt_\infty = 5 \rightarrow 5 = Nt_\infty$

$\Rightarrow n = (Id)$.

And so a symmetric representation for α is given by

$\alpha = t_3 t_\infty$.

Proof of Isomorphism

Double coset enumeration as displayed in Figure 7.1 shows that the group defined by the symmetric presentation contains a homomorphic image $N \cong L_2(5)$ to index at most

$$\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(0\ 1)}|} + \frac{|N|}{|N^{(0\ 1\ 0)}|} + \frac{|N|}{|N^{(0\ 1\ 2)}|} +$$

$$\frac{|N|}{|N^{(0\ 1\ 3)}|} + \frac{|N|}{|N^{(0\ 1\ 2\ 0)}|} + \frac{|N|}{|N^{(0\ 1\ 2\ 4)}|} + \frac{|N|}{|N^{(0\ 1\ 2\ \infty)}|} +$$

$$\frac{|N|}{|N^{(0\ 1\ 0\ 1)}|} + \frac{|N|}{|N^{(0\ 1\ 0\ 2)}|} + \frac{|N|}{|N^{(0\ 1\ 0\ 3)}|} + \frac{|N|}{|N^{(0\ 1\ 3\ 4)}|} +$$

$$\frac{|N|}{|N^{(0\ 1\ 2\ \infty\ 3)}|} + \frac{|N|}{|N^{(0\ 1\ 2\ \infty\ 4)}|} + \frac{|N|}{|N^{(0\ 1\ 0\ 1\ 0)}|}$$

$$= \frac{60}{60} + \frac{60}{10} + \frac{60}{2} + \frac{60}{2} + \frac{60}{1} + \frac{60}{1} + \frac{60}{5} + \frac{60}{1} + \frac{60}{1} +$$

$$\frac{60}{2} + \frac{60}{1} + \frac{60}{1} + \frac{60}{1} + \frac{60}{3} + \frac{60}{3} + \frac{60}{12}$$

$$= 1 + 6 + 30 + 30 + 60 + 60 + 12 + 60 + 60 + 30 + 60 + 60 +$$

$$60 + 20 + 20 + 5$$

$$= 574.$$

As a result, the index of N in G is at most 574.

Therefore, the order of the image group G is at most

$$574 \cdot |N| = 574 \cdot 60 = 34440$$

Furthermore, since the action of both N an the symmetric generators on the cosets are well defined, it suffices to show that

(1) t_0 has six images, under conjugation by N

(2) the six relations hold true within the symmetric group S_{574}.

Moreover, the action by right multiplication on the 574 cosets is implicit in the enumeration, and we can easily

verify that the image is the projective special linear group $L_2(41)$.

We define linear fractional permutations of the projective line $L_2(41)$ as follows:

x: $\left(\tau \mapsto \frac{39\tau + 22}{\tau - 32}\right)$ ≡ (1, 2, 24, 34, 18)(3, 15, 27, 31, 40)(4, 20, 22, 35, 25)(5, 36, 8, 10, 26)(6, 28, 29, 12, 37)(7, 21, 13, 11, 41)(9, 23, 30, 19, 17)(32, 42, 39, 33, 38)

y: $\left(\tau \mapsto \frac{14\tau + 7}{\tau - 32}\right)$ ≡ (1, 39, 38)(2, 33, 18)(3, 28, 13)(4, 8, 7)(5, 26, 27)(6, 17, 11)(9, 37, 23)(10, 25, 31)(12, 22, 30)(14, 32, 42)(15, 21, 36)(16, 24, 34)(19, 20, 41)(29, 40, 35)

t_0: $\left(\tau \mapsto \frac{28\tau + 11}{\tau - 28}\right)$ ≡ (1, 35)(2, 40)(3, 29)(4, 41)(5, 38)(6, 31)(7, 37)(8, 19)(9, 25)(10, 18)(11, 15)(12, 27)(13, 16)(14, 21)(17, 34)(20, 26)(22, 39)(23, 33)(28, 42)(30, 36)

We note that the order of the product xy equals 2, and so $N = \langle x, y \rangle = L_2(5)$, a maximal subgroup of $L_2(41)$.

The permutation t_0 has six images under conjugation by N, namely:

t_0 = (1, 35)(2, 40)(3, 29)(4, 41)(5, 38)(6, 31)(7, 37)(8, 19)(9, 25)(10, 18)(11, 15)(12, 27)(13, 16)(14, 21)(17, 34)(20, 26)(22, 39)(23, 33)(28, 42)(30, 36)

t_1 = (1, 26)(2, 25)(3, 24)(4, 23)(5, 22)(6, 21)(7, 20)(8, 19)(9,18)(10,17)(11,16)(12, 15)(13, 14)(27, 41)(28, 40)(29, 39)(30, 38)(31, 37)(32, 36)(33,35)

t_2 = (1, 23)(2, 5)(3, 29)(4, 24)(6, 40)(7, 31)(8, 42)(9, 26)(10, 17)(11, 14)(12, 33)(13, 28)(15, 34)(16, 41)(19, 32)(20, 30)(21, 22)(25, 38)(27, 37)(35,36)

t_3 = (2, 30)(3, 28)(4, 32)(5, 23)(6, 31)(7, 16)(8, 25)(9, 26)(10, 39)(11,29)(12,15)(13, 35)(14, 41)(17, 42)(18, 27)(19, 22)(20, 34)(21, 40)(24, 36)(37,38)

t_4 = (1, 31)(3, 13)(4, 10)(5, 23)(6, 32)(7, 14)(8, 34)(9, 39)(11, 25)(12, 41)(15, 29)(16, 21)(17, 35)(18, 22)(19, 24)(20, 42)(26, 33)(27, 37)(28, 40)(30, 36)

t_∞ = (1, 12)(2, 37)(3, 32)(4, 41)(5, 19)(6, 24)(7, 20)(8, 9)(10, 23)(11, 25)(13, 35)(15, 42)(17, 36)(18, 29)(21, 22)(26, 30)(27, 39)(28, 34)(31, 33)(38,40)

As a result, the action of x and y on $\{t_0, t_1, t_2, t_3, t_4, t_\infty\}$ is given by:

x: $(t_0, t_1, t_2, t_3, t_4)$

y: $(t_\infty, t_0, t_1)(t_2, t_4, t_3)$

Now, we need to verify that our relations hold true in $L_2(41)$.

Looking at our first relation we can see that

$t_0^{(t_1t_0t_4t_3t_2t_1t_0)}$ = (2, 30)(3, 28)(4, 32)(5, 23)(6, 31)(7, 16)(8, 25)(9, 26)(10, 39)(11, 29)(12, 15)(13, 35)(14, 41)(17, 42)(18, 27)(19, 22)(20, 34)(21, 40)(24, 36)(37, 38)

= t_3

$t_1^{(t_1t_0t_4t_3t_2t_1t_0)}$ = (1, 31)(3, 13)(4, 10)(5, 23)(6, 32)(7, 14)(8, 34)(9, 39)(11, 25)(12, 41)(15, 29)(16, 21)(17, 35)(18, 22)(19, 24)(20, 42)(26, 33)(27, 37)(28, 40)(30, 36)

= t_4

$t_2^{(t_1t_0t_4t_3t_2t_1t_0)}$ = (1, 35)(2, 40)(3, 29)(4, 41)(5, 38)(6, 31)(7, 37)(8, 19)(9, 25)(10, 18)(11, 15)(12, 27)(13, 16)(14, 21)(17, 34)(20, 26)(22, 39)(23, 33)(28, 42)(30, 36)

= t_0

$t_3^{(t_1t_0t_4t_3t_2t_1t_0)}$ = (1, 26)(2, 25)(3, 24)(4, 23)(5, 22)(6, 21)(7, 20)(8, 19)(9, 18)(10, 17)(11, 16)(12, 15)(13, 14)(27, 41)(28, 40)(29, 39)(30, 38)(31, 37)

$(32, 36)(33, 35)$

$= t_1$

$t_4^{(t_1t_0t_4t_3t_2t_1t_0)} = (1, 23)(2, 5)(3, 29)(4, 24)(6, 40)(7, 31)(8, 42)(9, 26)(10, 17)(11, 14)(12, 33)(13, 28)(15, 34)(16, 41)(19, 32)(20, 30)(21, 22)(25, 38)(27, 37)(35, 36)$

$= t_2$

$t_\infty^{(t_1t_0t_4t_3t_2t_1t_0)} = (1, 12)(2, 37)(3, 32)(4, 41)(5, 19)(6, 24)(7, 20)(8, 9)(10, 23)(11, 25)(13, 35)(15, 42)(17, 36)(18, 29)(21, 22)(26, 30)(27, 39)(28, 34)(31, 33)(38, 40)$

$= t_\infty.$

Thus, $t_1t_0t_4t_3t_2t_1t_0$ acts as the permutation (1 4 2 0 3 1) on the symmetric generators, that is

$$t_1t_0t_4t_3t_2t_1t_0 = (1\ 4\ 2\ 0\ 3)$$

$$\Rightarrow t_0t_1t_2t_3 = (1\ 3\ 0\ 2\ 4)t_1t_0t_4.$$

Therefore, our first relation holds true in $L_2(41)$.

Similarly, looking at our second relation we can see that

$t_0^{(t_\infty t_1t_0t_\infty t_1t_0t_\infty)} = (1, 12)(2, 37)(3, 32)(4, 41)(5, 19)(6, 24)(7, 20)(8, 9)(10, 23)(11, 25)(13, 35)(15, 42)(17, 36)(18, 29)(21, 22)(26, 30)(27, 39)(28, 34)(31, 33)(38, 40)$

$= t_\infty$

$t_1^{(t_\infty t_1 t_0 t_\infty t_1 t_0 t_\infty)} = (1, 35)(2, 40)(3, 29)(4, 41)(5, 38)(6, 31)(7, 37)(8, 19)(9, 25)(10, 18)(11, 15)(12, 27)(13, 16)(14, 21)(17, 34)(20, 26)(22, 39)(23, 33)(28, 42)(30, 36)$

$= t_0$

$t_2^{(t_\infty t_1 t_0 t_\infty t_1 t_0 t_\infty)} = (2, 30)(3, 28)(4, 32)(5, 23)(6, 31)(7, 16)(8, 25)(9, 26)(10, 39)(11, 29)(12, 15)(13, 35)(14, 41)(17, 42)(18, 27)(19, 22)(20, 34)(21, 40)(24, 36)(37, 38)$

$= t_3$

$t_3^{(t_\infty t_1 t_0 t_\infty t_1 t_0 t_\infty)} = (1, 31)(3, 13)(4, 10)(5, 23)(6, 32)(7, 14)(8, 34)(9, 39)(11, 25)(12, 41)(15, 29)(16, 21)(17, 35)(18, 22)(19, 24)(20, 42)(26, 33)(27, 37)(28, 40)(30, 36)$

$= t_4$

$t_4^{(t_\infty t_1 t_0 t_\infty t_1 t_0 t_\infty)} = (1, 23)(2, 5)(3, 29)(4, 24)(6, 40)(7, 31)(8, 42)(9, 26)(10, 17)(11, 14)(12, 33)(13, 28)(15, 34)(16, 41)(19, 32)(20, 30)(21, 22)(25, 38)(27, 37)(35, 36)$

$= t_2$

$t_\infty^{(t_\infty t_1 t_0 t_\infty t_1 t_0 t_\infty)}$ = (1, 26)(2, 25)(3, 24)(4, 23)(5, 22)(6, 21)(7, 20)(8, 19)(9, 18)(10, 17)(11, 16)(12, 15)(13, 14)(27, 41)(28, 40)(29, 39)(30, 38)(31, 37)(32, 36)(33, 35)

= t_1.

Thus, $t_\infty t_1 t_0 t_\infty t_1 t_0 t_\infty$ acts as the permutation (1 0 ∞)(2 3 4) on the symmetric generators, that is

$$t_\infty t_1 t_0 t_\infty t_1 t_0 t_\infty = (1\ 0\ \infty)(2\ 3\ 4)$$

$$\Rightarrow (1\ \infty\ 0)(2\ 4\ 3) t_\infty t_1 t_0 t_\infty = t_\infty t_0 t_1.$$

Therefore, our second relation holds true in $L_2(41)$.

Looking at our third relation we can see that

$t_0^{(t_0 t_1 t_2 t_0 t_4 t_1 t_0 t_4)}$ = (1, 12)(2, 37)(3, 32)(4, 41)(5, 19)(6, 24)(7, 20)(8, 9)(10, 23)(11, 25)(13, 35)(15, 42)(17, 36)(18, 29)(21, 22)(26, 30)(27, 39)(28, 34)(31, 33)(38, 40)

= t_∞

$t_1^{(t_0 t_1 t_2 t_0 t_4 t_1 t_0 t_4)}$ = (1, 26)(2, 25)(3, 24)(4, 23)(5, 22)(6, 21)(7, 20)(8, 19)(9, 18)(10, 17)(11, 16)(12, 15)(13, 14)(27, 41)(28, 40)(29, 39)(30, 38)(31, 37)(32, 36)(33, 35)

= t_1

$t_2^{(t_0t_1t_2t_0t_4t_1t_0t_4)}$ = (1, 31)(3, 13)(4, 10)(5, 23)(6, 32)(7, 14)(8, 34)(9, 39)(11, 25)(12, 41)(15, 29)(16, 21)(17, 35)(18, 22)(19, 24)(20, 42)(26, 33)(27, 37)(28, 40)(30, 36)

= t_4

$t_3^{(t_0t_1t_2t_0t_4t_1t_0t_4)}$ = (1, 35)(2, 40)(3, 29)(4, 41)(5, 38)(6, 31)(7, 37)(8, 19)(9, 25)(10, 18)(11, 15)(12, 27)(13, 16)(14, 21)(17, 34)(20, 26)(22, 39)(23, 33)(28, 42)(30, 36)

= t_0

$t_4^{(t_0t_1t_2t_0t_4t_1t_0t_4)}$ = (2, 30)(3, 28)(4, 32)(5, 23)(6, 31)(7, 16)(8, 25)(9, 26)(10, 39)(11, 29)(12, 15)(13, 35)(14, 41)(17, 42)(18, 27)(19, 22)(20, 34)(21, 40)(24, 36)(37, 38)

= t_3

$t_\infty^{(t_0t_1t_2t_0t_4t_1t_0t_4)}$ = (1, 23)(2, 5)(3, 29)(4, 24)(6, 40)(7, 31)(8, 42)(9, 26)(10, 17)(11, 14)(12, 33)(13, 28)(15, 34)(16, 41)(19, 32)(20, 30)(21, 22)(25, 38)(27, 37)(35, 36)

= t_2.

Thus, $t_0t_1t_2t_0t_4t_1t_0t_4$ acts as the permutation (0 ∞ 2 4 3) on the symmetric generators, that is

$$t_0t_1t_2t_0t_4t_1t_0t_4 = (0\ \infty\ 2\ 4\ 3)$$

$$\Rightarrow t_0t_1t_2t_0 = (0\ \infty\ 2\ 4\ 3)t_4t_0t_1t_4.$$

Therefore, our third relation holds true in $L_2(41)$.

Now, looking at our fourth relation we can see that

$$t_0^{(t_0t_1t_2t_\infty t_3t_4t_2t_1t_\infty t_3)} = (1,\ 35)(2,\ 40)(3,\ 29)(4,\ 41)(5,\ 38)(6,\ 31)(7,\ 37)(8,\ 19)(9,\ 25)(10,\ 18)(11,\ 15)(12,\ 27)(13,\ 16)(14,\ 21)(17,\ 34)(20,\ 26)(22,\ 39)(23,\ 33)(28,\ 42)(30,\ 36)$$

$$= t_0$$

$$t_1^{(t_0t_1t_2t_\infty t_3t_4t_2t_1t_\infty t_3)} = (1,\ 26)(2,\ 25)(3,\ 24)(4,\ 23)(5,\ 22)(6,\ 21)(7,\ 20)(8,\ 19)(9,\ 18)(10,\ 17)(11,\ 16)(12,\ 15)(13,\ 14)(27,\ 41)(28,\ 40)(29,\ 39)(30,\ 38)(31,\ 37)(32,\ 36)(33,\ 35)$$

$$= t_1$$

$$t_2^{(t_0t_1t_2t_\infty t_3t_4t_2t_1t_\infty t_3)} = (1,\ 31)(3,\ 13)(4,\ 10)(5,\ 23)(6,\ 32)(7,\ 14)(8,\ 34)(9,\ 39)(11,\ 25)(12,\ 41)(15,\ 29)(16,\ 21)(17,\ 35)(18,\ 22)(19,\ 24)(20,\ 42)(26,\ 33)(27,\ 37)(28,\ 40)(30,\ 36)$$

$$= t_4$$

$$t_3^{(t_0t_1t_2t_\infty t_3t_4t_2t_1t_\infty t_3)} = (1,\ 12)(2,\ 37)(3,\ 32)(4,\ 41)(5,\ 19)(6,\ 24)(7,\ 20)(8,\ 9)(10,\ 23)(11,\ 25)(13,\ 35)(15,\ 42)(17,\ 36)(18,\ 29)(21,\ 22)(26,\ 30)(27,$$

$$= (39)(28, 34)(31, 33)(38, 40)$$

$$= t_\infty$$

$$t_4^{(t_0t_1t_2t_\infty t_3t_4t_2t_1t_\infty t_3)} = (1, 23)(2, 5)(3, 29)(4, 24)(6, 40)(7, 31)(8, 42)(9, 26)(10, 17)(11, 14)(12, 33)(13, 28)(15, 34)(16, 41)(19, 32)(20, 30)(21, 22)(25, 38)(27, 37)(35, 36)$$

$$= t_2$$

$$t_\infty^{(t_0t_1t_2t_\infty t_3t_4t_2t_1t_\infty t_3)} = (2, 30)(3, 28)(4, 32)(5, 23)(6, 31)(7, 16)(8, 25)(9, 26)(10, 39)(11, 29)(12, 15)(13, 35)(14, 41)(17, 42)(18, 27)(19, 22)(20, 34)(21, 40)(24, 36)(37, 38)$$

$$= t_3.$$

Thus, $t_0t_1t_2t_\infty t_3t_4t_2t_1t_\infty t_3$ acts as the permutation $(2\ 4)(3\ \infty)$ on the symmetric generators, that is

$$t_0t_1t_2t_\infty t_3t_4t_2t_1t_\infty t_3 = (2\ 4)(3\ \infty)$$

$$\Rightarrow t_0t_1t_2t_\infty t_3 = (2\ 4)(3\ \infty)t_3t_\infty t_1t_2t_4.$$

Therefore, our fourth relation holds true in $L_2(41)$.

Now, looking at our fifth relation we can see that

$$t_0^{(t_0t_1t_2t_\infty t_4t_2t_1t_0t_3t_4)} = (1, 35)(2, 40)(3, 29)(4, 41)(5, 38)(6, 31)(7, 37)(8, 19)(9, 25)(10, 18)(11, 15)(12, 27)(13, 16)(14, 21)(17, 34)(20, 26)(22, 39)(23, 33)(28, 42)(30, 36)$$

$= t_0$

$t_1^{(t_0t_1t_2t_\infty t_4t_2t_1t_0t_3t_4)} = (1, 31)(3, 13)(4, 10)(5, 23)(6, 32)(7, 14)$
$(8, 34)(9, 39)(11, 25)(12, 41)(15, 29)(16,$
$21)(17, 35)(18, 22)(19, 24)(20, 42)(26,$
$33)(27, 37)(28, 40)(30, 36)$

$= t_4$

$t_2^{(t_0t_1t_2t_\infty t_4t_2t_1t_0t_3t_4)} = (2, 30)(3, 28)(4, 32)(5, 23)(6, 31)(7, 16)$
$(8, 25)(9, 26)(10, 39)(11, 29)(12, 15)(13,$
$35)(14, 41)(17, 42)(18, 27)(19, 22)(20,$
$34)(21, 40)(24, 36)(37, 38)$

$= t_3$

$t_3^{(t_0t_1t_2t_\infty t_4t_2t_1t_0t_3t_4)} = (1, 23)(2, 5)(3, 29)(4, 24)(6, 40)(7, 31)$
$(8, 42)(9, 26)(10, 17)(11, 14)(12, 33)(13,$
$28)(15, 34)(16, 41)(19, 32)(20, 30)(21,$
$22)(25, 38)(27, 37)(35, 36)$

$= t_2$

$t_4^{(t_0t_1t_2t_\infty t_4t_2t_1t_0t_3t_4)} = (1, 26)(2, 25)(3, 24)(4, 23)(5, 22)(6, 21)$
$(7, 20)(8, 19)(9, 18)(10, 17)(11, 16)(12,$
$15)(13, 14)(27, 41)(28, 40)(29, 39)(30,$
$38)(31, 37)(32, 36)(33, 35)$

$= t_1$

$t_\infty^{(t_0t_1t_2t_\infty t_4t_2t_1t_0t_3t_4)}$ = (1, 12)(2, 37)(3, 32)(4, 41)(5, 19)(6, 24)
(7, 20)(8, 9)(10, 23)(11, 25)(13, 35)(15,
42)(17, 36)(18, 29)(21, 22)(26, 30)(27,
39)(28, 34)(31, 33)(38, 40)

$= t_\infty$.

Thus, $t_0t_1t_2t_\infty t_4t_2t_1t_0t_3t_4$ acts as the permutation (1 4)(2 3) on the symmetric generators, that is

$$t_0t_1t_2t_\infty t_3t_4t_2t_1t_\infty t_3 = (1\ 4)(2\ 3)$$

$$\Rightarrow t_0t_1t_2t_\infty t_4 = (1\ 4)(2\ 3)t_4t_3t_0t_1t_2.$$

Therefore, our fifth relation holds true in $L_2(41)$.

Now, looking at our sixth relation we can see that

$t_0^{(t_0t_1t_0t_1t_0t_3t_\infty t_3t_\infty t_3)}$ = (1, 23)(2, 5)(3, 29)(4, 24)(6, 40)(7, 31)
(8, 42)(9, 26)(10, 17)(11, 14)(12, 33)(13,
28)(15, 34)(16, 41)(19, 32)(20, 30)(21,
22)(25, 38)(27, 37)(35, 36)

$= t_2$

$t_1^{(t_0t_1t_0t_1t_0t_3t_\infty t_3t_\infty t_3)}$ = (1, 31)(3, 13)(4, 10)(5, 23)(6, 32)(7, 14)
(8, 34)(9, 39)(11, 25)(12, 41)(15, 29)(16,
21)(17, 35)(18, 22)(19, 24)(20, 42)(26,
33)(27, 37)(28, 40)(30, 36)

$= t_4$

$t_2^{(t_0t_1t_0t_1t_0t_3t_\infty t_3t_\infty t_3)}$ = (2, 30)(3, 28)(4, 32)(5, 23)(6, 31)(7, 16)

$$(8,\ 25)(9,\ 26)(10,\ 39)(11,\ 29)(12,\ 15)(13,\ 35)(14,\ 41)(17,\ 42)(18,\ 27)(19,\ 22)(20,\ 34)(21,\ 40)(24,\ 36)(37,\ 38)$$

$$= t_3$$

$$t_3^{(t_0t_1t_0t_1t_0t_3t_\infty t_3t_\infty t_3)} = (1,\ 35)(2,\ 40)(3,\ 29)(4,\ 41)(5,\ 38)(6,\ 31)(7,\ 37)(8,\ 19)(9,\ 25)(10,\ 18)(11,\ 15)(12,\ 27)(13,\ 16)(14,\ 21)(17,\ 34)(20,\ 26)(22,\ 39)(23,\ 33)(28,\ 42)(30,\ 36)$$

$$= t_0$$

$$t_4^{(t_0t_1t_0t_1t_0t_3t_\infty t_3t_\infty t_3)} = (1,\ 12)(2,\ 37)(3,\ 32)(4,\ 41)(5,\ 19)(6,\ 24)(7,\ 20)(8,\ 9)(10,\ 23)(11,\ 25)(13,\ 35)(15,\ 42)(17,\ 36)(18,\ 29)(21,\ 22)(26,\ 30)(27,\ 39)(28,\ 34)(31,\ 33)(38,\ 40)$$

$$= t_\infty$$

$$t_\infty^{(t_0t_1t_0t_1t_0t_3t_\infty t_3t_\infty t_3)} = (1,\ 26)(2,\ 25)(3,\ 24)(4,\ 23)(5,\ 22)(6,\ 21)(7,\ 20)(8,\ 19)(9,\ 18)(10,\ 17)(11,\ 16)(12,\ 15)(13,\ 14)(27,\ 41)(28,\ 40)(29,\ 39)(30,\ 38)(31,\ 37)(32,\ 36)(33,\ 35)$$

$$= t_1.$$

Thus, $t_0t_1t_0t_1t_0t_3t_\infty t_3t_\infty t_3$ acts as the permutation $(0\ 2\ 3)(1\ 4\ \infty)$ on the symmetric generators, that is

$$t_0t_1t_0t_1t_0t_3t_\infty t_3t_\infty t_3 = (0\ 2\ 3)(1\ 4\ \infty)$$

$$\Rightarrow t_0t_1t_0t_1t_0 = (0\ 2\ 3)(1\ 4\ \infty)t_3t_\infty t_3t_\infty t_3.$$

Therefore, our sixth relation holds true in $L_2(41)$.

Since our relations hold in $L_2(41)$, N is maximal in $L_2(41)$ and $t_0 \notin N$, where t_0 has 6 conjugates under conjugation by N, then $L_2(41)$ is generated by N and t_0. Therefore, $L_2(41)$ is an image of G. As a result, $|G| \geq |L_2(41)| = 34440$. But from previous we know that $|G| \leq 34440$, and so

$$|G| \leq 34440 = |L_2(41)| \leq |G|,$$

which proves the isomorphism. Hence, $G \cong L_2(41)$.

CHAPTER EIGHT

NON-SYMMETRICALLY GENERATED GROUPS

Introduction

In this chapter we will give examples of groups that are not symmetrically generated by the involutions, t_i's.

Group $2^4 : (A_5 \times 2)$

Factoring the progenitor $2^{*5} : A_5$ by the following relations

$$[(0\ 1\ 2)t_0]^6 \text{ and } [t_0t_1]^2$$

yields the finite homomorphic image:

$$G \cong 2^4 : (A_5 \times 2) \cong \frac{2^{*5} : A_5}{[(0\ 1\ 2)t_0]^6,\ [t_0t_1]^2}.$$

A symmetric presentation for the above image is given by:

<x,y,z,t|x^3, y^3, z^3, (x*y)^2, (x*z)^2,

(y*z)^2, t^2, (t,x), (t,y), (z*t)^6, (t*t^z)^2>,

where N = A_5 ≅ <x,y,z|x^3, y^3, z^3, (x*y)^2, (x*z)^2,

(y*z)^2>,

and the action of x, y, and z on the symmetric generators is given by

x ~ (1 2 3)

y ~ (1 2 4)

z ~ (1 2 0).

Double Coset Enumeration

Before looking at any relations, it should be mentioned that G will not be symmetrically generated by $\{t_0, t_1, t_2, t_3\}$. As a matter of fact, the set of involutions as mentioned before will only generate a subgroup of G of index 60.

Expanding the relation $[t_0t_1]^2 = 1$ yields:

$t_0t_1t_0t_1 = 1$

$\Rightarrow \boxed{t_0t_1 = t_1t_0 \qquad (1)}$

$\Rightarrow t_it_j = t_jt_i.$

Therefore, $t_0t_1 = t_1t_0$.

In like manner, we have

$t_0t_2 = t_2t_0$	$t_1t_2 = t_2t_1$	$t_2t_3 = t_3t_2$
$t_0t_3 = t_3t_0$	$t_1t_3 = t_3t_1$	$t_2t_4 = t_4t_2$
$t_0t_4 = t_4t_0$	$t_1t_4 = t_4t_1$	$t_3t_4 = t_4t_3$.

So, the double coset [0 1] has $5 \cdot 4 = 20$ single cosets. But since t_0t_1 has two names, then [0 1] only has $20/2 = 10$ distinct single cosets.

Furthermore, applying t_0 to $t_0t_1 = t_1t_0$ yields:

$t_0t_1t_0 = t_1t_0t_0$

$= t_1$

but $t_1 \in [0]$, (since $t_1^{\pi} = t_0$, where $\pi = (1\ 0\ 2) \in N$).

Therefore, [0 1 0] = [0]. Since $N^{(0\ 1)}$ has orbits {0, 1} and {2, 3, 4} then [0 1 i] = [0] for all i in {0, 1}.

Now, expanding the relation $[(0\ 1\ 2)t_0]^6 = 1$ yields:

$\pi^6(t_0)^{\pi^5}(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}(t_0) = 1$, where $\pi = (0\ 1\ 2) \in N$

$\Rightarrow (Id)\, t_2t_1t_0t_2t_1t_0 = 1$

$\Rightarrow \boxed{t_0t_1t_2 = t_2t_1t_0.} \quad (2)$

Now, from (1) we have

$t_0t_1 = t_1t_0$

$\Rightarrow t_0t_1t_2 = t_1t_0t_2.$

And from (2) we have

$t_0t_1t_2 = \underline{t_2t_1}t_0$

$= t_1t_2t_0$ (since $t_2t_1 = t_1t_2$),

and

$t_2t_1t_0 = t_0\underline{t_1t_2}$

$= \underline{t_0t_2}t_1$ (since $t_1t_2 = t_2t_1$)

$= t_2t_0t_1$ (since $t_0t_2 = t_2t_0$).

Therefore,

$t_0t_1t_2 = t_2t_0t_1 = t_1t_0t_2 = t_0t_2t_1 = t_1t_2t_0 = t_2t_1t_0.$

In like manner, we have

$t_0t_1t_3 = t_3t_0t_1 = t_1t_0t_3 = t_0t_3t_1 = t_1t_3t_0 = t_3t_1t_0$

$t_0t_1t_4 = t_4t_0t_1 = t_1t_0t_4 = t_0t_4t_1 = t_1t_4t_0 = t_4t_1t_0$

$t_0t_2t_3 = t_3t_0t_2 = t_2t_0t_3 = t_0t_3t_2 = t_2t_3t_0 = t_3t_2t_0$

$t_0t_2t_4 = t_4t_0t_2 = t_2t_0t_4 = t_0t_4t_2 = t_2t_4t_0 = t_4t_2t_0$

$t_0t_3t_4 = t_4t_0t_3 = t_3t_0t_4 = t_0t_4t_3 = t_3t_4t_0 = t_4t_3t_0$

$t_1t_2t_3 = t_3t_1t_2 = t_2t_1t_3 = t_1t_3t_2 = t_2t_3t_1 = t_3t_2t_1$

$t_1t_2t_4 = t_4t_1t_2 = t_2t_4t_1 = t_4t_2t_1 = t_1t_4t_2 = t_2t_1t_4$

$t_1t_3t_4 = t_4t_1t_3 = t_3t_4t_1 = t_4t_3t_1 = t_1t_4t_3 = t_3t_1t_4$

$t_2t_3t_4 = t_4t_2t_3 = t_3t_4t_2 = t_4t_3t_2 = t_2t_4t_3 = t_3t_2t_4$.

Therefore, the double coset [0 1 2] has 10 distinct single cosets, each with 6 names.

Furthermore, applying t_0 to (2) yields:

$t_0t_1t_2t_0 = t_2t_1t_0t_0$

$= t_2t_1$

but $t_2t_1 \in [0\ 1]$, (since $(t_2t_1)^{\pi} = t_0t_1$,

where $\pi = (2\ 0\ 3) \in N$).

Therefore, [0 1 2 0] = [0 1]. Since $N^{(0\ 1\ 2)}$ has orbits {0, 1, 2} and {3, 4} then [0 1 2 j] = [0 1] for all j in {0, 1, 2}.

Now, since $t_0t_1t_2 = t_1t_2t_0$

$\Rightarrow t_0t_1t_2t_3 = t_1\underline{t_2t_0t_3}$

$= t_1\underline{t_3t_2t_0}$ (since $t_2t_0t_3 = t_3t_2t_0$)

$= \underline{t_1t_0t_3}t_2$ (since $t_3t_2t_0 = t_0t_3t_2$)

$= \underline{t_0t_3t_1}t_2$ (since $t_1t_0t_3 = t_0t_3t_1$)

$= t_3\underline{t_1t_0t_2}$ (since $t_0t_3t_1 = t_3t_1t_0$)

$= t_3\underline{t_2t_1t_0}$ (since $t_1t_0t_2 = t_2t_1t_0$)

$= \underline{t_3t_0t_2}t_1$ (since $t_2t_1t_0 = t_0t_2t_1$)

$= \underline{t_2t_3t_0}t_1$ (since $t_3t_0t_2 = t_2t_3t_0$)

$= t_0t_2t_3t_1$ (since $t_2t_3t_0 = t_0t_2t_3$)

and

$t_1t_2t_0t_3 = \underline{t_0t_1t_2}t_3$

$= t_2\underline{t_0t_1t_3}$ (since $t_0t_1t_2 = t_2t_0t_1$)

$= t_2t_1t_3t_0$ (since $t_0t_1t_3 = t_1t_3t_0$).

Therefore,

$t_0t_1t_2t_3 = t_1t_2t_0t_3 = t_1t_3t_2t_0 = t_1t_0t_3t_2 = t_0t_3t_1t_2 = t_3t_1t_0t_2 =$

$t_3t_2t_1t_0 = t_3t_0t_2t_1 = t_2t_3t_0t_1 = t_0t_2t_3t_1 = t_2t_0t_1t_3 = t_2t_1t_3t_0.$

So the double coset [0 1 2 3] has 5 distinct single cosets, namely $Nt_0t_1t_2t_3$, $Nt_0t_1t_2t_4$, $Nt_0t_1t_3t_4$, $Nt_0t_2t_3t_4$, $Nt_1t_2t_3t_4$, each with 12 names.

Furthermore, applying t_0 to $t_0t_1t_2t_3 = t_1t_3t_2t_0$ yields:

$t_0t_1t_2t_3t_0 = t_1t_3t_2t_0t_0$

$= t_1t_3t_2$

but $t_1t_3t_2 \in$ [0 1 2] (since $(t_1t_3t_2)^{\pi} = t_0t_1t_2$,

where $\pi = (0\ 3\ 1) \in N$).

Therefore, [0 1 2 3 0] = [0 1 2]. Since $N^{(0\ 1\ 2\ 3)}$ has orbits {4} and {0, 1, 2, 3} then [0 1 2 3 k] = [0 1 2] for all k in {0, 1, 2, 3}.

Now, applying t_4 to $t_0t_1t_2t_3 = t_1t_2t_0t_3$ yields:

$\Rightarrow t_0t_1t_2t_3t_4 = t_1t_2t_0t_3t_4$

$\Rightarrow t_0t_1t_2t_3t_4 = t_1t_2t_3t_4t_0$.

By using the above relation to calculate the order of the stabilizing group $N^{(0\ 1\ 2\ 3\ 4)}$ (see Appendix C(1)) we can see that its order equals 60. Therefore, the number of distinct single cosets in the double coset [0 1 2 3 4] turns out to be $\frac{|N|}{|N^{(0\ 1\ 2\ 3\ 4)}|} = \frac{60}{60} = 1$, which implies that all the single cosets in [0 1 2 3 4] are all equal to each other.

Furthermore, $N^{(0\ 1\ 2\ 3\ 4)}$ is transitive, which means that it has orbit {0, 1, 2, 3, 4}. As a result [0 1 2 3 4 m] = [0 1 2 3] for all m in {0, 1, 2, 3, 4}.

Therefore, we must have found all the double cosets as illustrated in Figure 8.1.

The labeling used for the single cosets of N in G is given by:

Label	Single Coset
32.	[N]
1.	[1]
2.	[2]
3.	[3]
4.	[4]
5.	[0]
6.	[0 1]
7.	[0 2]
8.	[0 3]
9.	[0 4]
10.	[1 2]

11. [1 3]
12. [1 4]
13. [2 3]
14. [2 4]
15. [3 4]
16. [0 1 2]
17. [0 1 3]
18. [0 1 4]
19. [0 2 3]
20. [0 2 4]
21. [0 3 4]
22. [1 2 3]
23. [1 2 4]
24. [1 3 4]
25. [2 3 4]
26. [0 1 2 3]
27. [0 1 2 4]
28. [0 1 3 4]
29. [0 2 3 4]
30. [1 2 3 4]
31. [0 1 2 3 4]

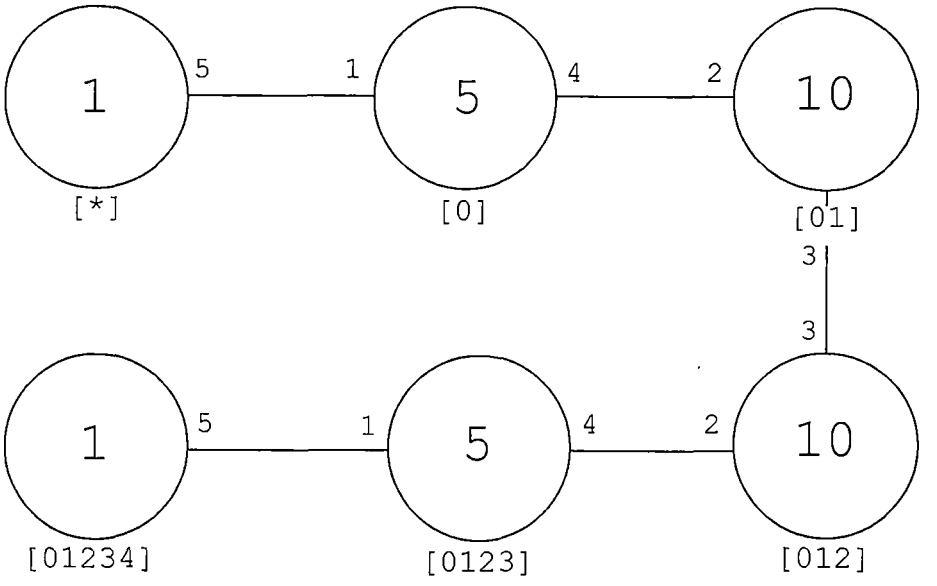

Figure 8.1 Cayley Graph of 2^4 : (A_5 x 2) over A_5

The action of the symmetric generators on the cosets of N in G is given by:

t_0 = (32,5)(1,6)(2,7)(3,8)(4,9)(10,16)(11,17)(12,18)(13,19)
(14,20)(15,21)(22,26)(23,27)(24,28)(25,29)(30,31)

t_1 = (32,1)(5,6)(2,10)(3,11)(4,12)(7,16)(8,17)(9,18)(13,22)
(14,23)(15,24)(19,26)(20,27)(21,28)(25,30)(29,31)

t_2 = (32,2)(5,7)(1,10)(3,13)(4,14)(6,16)(8,19)(9,20)(11,22)
(12,23)(15,25)(17,26)(18,27)(21,29)(24,30)(28,31)

t_3 = (32,3)(5,8)(1,11)(2,13)(4,15)(6,17)(7,19)(9,21)(10,22)
(12,24)(14,25)(16,26)(18,28)(20,29)(23,30)(27,31)

t_4 = (32,4)(5,9)(1,12)(2,14)(3,15)(6,18)(7,20)(8,21)(10,23)
(11,24)(13,25)(16,27)(17,28)(19,29)(22,30)(26,31)

Group $2 \times S_5$

Factoring the progenitor $2^{*4} : S_4$ by the following relation

$$[(0\ 1\ 2)t_0]^4$$

yields the finite homomorphic image:

$$G \cong (2 \times S_5) = \frac{2^{*4} : S_4}{[(0\ 1\ 2)t_0]^4}.$$

A symmetric presentation for the above image is given by:

<x,y,t|x^4, y^2, (y*x)^3, t^2, (t,y),

(t,(x*y)^(y*y^(y*x))), (x*t)^4>,

where $N = S_4 \cong$ <x,y|x^4, y^2, (y*x)^3>,

and the action of x and y on the symmetric generators is given by

$x \sim (0\ 1\ 2\ 3)$

$y \sim (2\ 3)$.

Double Coset Enumeration

Before looking at any relations, it should be mentioned that G will not be symmetrically generated by $\{t_0, t_1, t_2, t_3\}$. As a matter of fact, the set of involutions as mentioned before will only generate a subgroup of G of index 2.

Expanding the relation $[(0\ 1\ 2)t_0]^4$ yields:

$\pi^4(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}(t_0) = 1$, where $\pi = (0\ 1\ 2) \in N$

$\Rightarrow (0\ 1\ 2)t_0t_2t_1t_0 = 1$

$\Rightarrow \boxed{t_0t_1 = (0\ 1\ 2)t_0t_2 \quad (1)}$

$\Rightarrow Nt_0t_1 = Nt_0t_2$

$= Nt_0t_3$ (since $(t_0t_2)^{(2\ 3)} = t_0t_3$).

Therefore, $Nt_0t_1 = Nt_0t_2 = Nt_0t_3$.

In like manner, we have

$Nt_1t_0 = Nt_1t_2 = Nt_1t_3$

$Nt_2t_0 = Nt_2t_1 = Nt_2t_3$

$Nt_3t_0 = Nt_3t_1 = Nt_3t_2$

So, the double coset [0 1] has $4 \cdot 3 = 12$ single cosets. But since t_0t_1 has three names, then [0 1] only has $12/3 = 4$ distinct single cosets.

Furthermore, applying t_2 to (1) yields:

$t_0t_1t_2 = (0\ 1\ 2)t_0t_2t_2$

$= (0\ 1\ 2)t_0$

$\Rightarrow Nt_0t_1t_2 = Nt_0$.

Therefore, [0 1 2] = [0]. Since $N^{(0\ 1)}$ has orbits {0} and {1, 2, 3}, then [0 1 i] = [0] for all i in {1, 2, 3}.

Now, applying t_0 to (1) we have

$t_0t_1t_0 = (0\ 1\ 2)t_0t_2t_0$

$\Rightarrow Nt_0t_1t_0 = N\underline{t_0t_2}t_0$

$= Nt_0\underline{t_3t_0}$ (since $Nt_0t_2 = Nt_0t_3$)

$= Nt_0(3\ 0\ 2)t_3t_2$ (since $t_3t_0 = (3\ 0\ 2)t_3t_2$)

$= N\underline{t_2t_3}t_2$

$= N\underline{t_2t_1}t_2$ (since $Nt_2t_3 = Nt_2t_1$)

$= Nt_2\underline{t_0t_2}$ (since $Nt_2t_1 = Nt_2t_0$)

$= Nt_2(0\ 2\ 1)t_0t_1$ (since $t_0t_2 = (0\ 2\ 1)t_0t_1$)

$= N\underline{t_1t_0}t_1$

$= N\underline{t_1t_2}t_1$ (since $Nt_1t_0 = Nt_1t_2$)

$= Nt_1\underline{t_3t_1}$ (since $Nt_1t_2 = Nt_1t_3$)

$= Nt_1(3\ 1\ 2)t_3t_2$ (since $t_3t_1 = (3\ 1\ 2)t_3t_2$)

$= Nt_2t_3t_2$

$= Nt_2\underline{t_1t_2}$ (since $Nt_2t_3 = Nt_2t_0$)

$= Nt_2(3\ 0\ 2)t_3t_0$ (since $t_1t_2 = (3\ 0\ 2)t_3t_2$),

and

$Nt_0t_1t_0 = Nt_0\underline{t_2t_0}$

$= Nt_0(2\ 0\ 3)t_2t_3$ (since $t_2t_0 = (2\ 0\ 3)t_2t_3$)

$= N\underline{t_3t_2}t_3$

$= N\underline{t_3t_0}t_3$ (since $Nt_3t_2 = Nt_3t_0$)

$= N\underline{t_3t_1}t_3$ (since $Nt_3t_0 = Nt_3t_1$).

Therefore,

$Nt_0t_1t_0 = Nt_0t_2t_0 = Nt_0t_3t_0 = Nt_1t_0t_1 = Nt_1t_2t_1 = Nt_1t_3t_1 =$

$Nt_2t_0t_2 = Nt_2t_1t_2 = Nt_2t_3t_2 = Nt_3t_0t_3 = Nt_3t_1t_3 = Nt_3t_2t_3.$

So the double coset [0 1 0] has $4 \cdot 3 \cdot 1 = 12$ single cosets. But since $t_0t_1t_0$ has twelve names, then [0 1 0] only has $12/12 = 1$ distinct single coset.

Furthermore, applying t_2 to $Nt_0t_1t_0 = Nt_2t_0t_2$ yields:

$Nt_0t_1t_0t_2 = Nt_2t_0t_2t_2$

$= Nt_2t_0$

but $t_2t_0 \in [0\ 1]$ (since $(t_2t_0)^{\pi} = t_0t_1$,

where $\pi = (0\ 1\ 2) \in N$).

Therefore, [0 1 0 2] = [0 1]. Since $N^{(0\ 1\ 0)}$ has orbit {0, 1, 2, 3} then [0 1 0 j] = [0 1] for all j in {0, 1, 2, 3}, which implies that $N^{(0\ 1\ 0)}$ is transitive on $\{t_0, t_1, t_2, t_3\}$. Therefore, we must have found all the double cosets as illustrated in Figure 8.2.

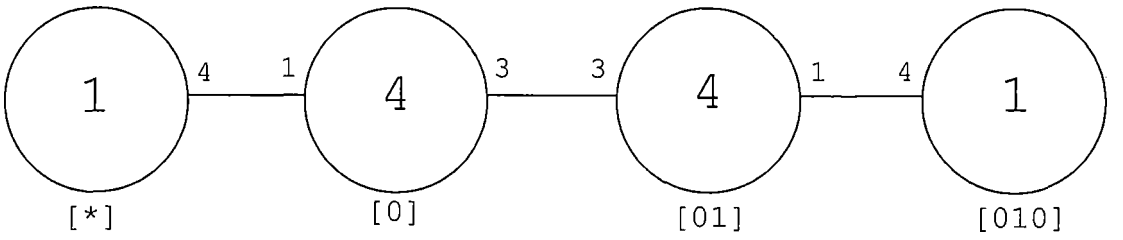

Figure 8.2 Cayley Graph of (2 x S_5) over S_4

The labeling used for the single cosets of N in G is given by:

Label	Single Coset
10.	[N]
1.	[1]
2.	[2]
3.	[3]
4.	[0]
5.	[0 1]
6.	[1 0]
7.	[2 0]
8.	[3 0]
9.	[0 1 0]

The action of the symmetric generators on the cosets of N in G is given by:

t_0 = (10,4)(1,6)(2,7)(3,8)(5,9)

t_1 = (10,1)(4,5)(2,7)(3,8)(6,9)

t_2 = (10,2)(4,5)(1,6)(3,8)(7,9)

t_3 = (10,3)(4,5)(1,6)(2,7)(8,9).

Group $2^4 : (S_5 \times 2)$

Factoring the progenitor $2^{*5} : S_5$ by the following relation

$$[(0\ 1)t_0]^4$$

yields the finite homomorphic image:

$$G \cong 2^4 : (S_5 \times 2) \cong \frac{2^{*5} : S_5}{[(0\ 1)t_0]^4}.$$

A symmetric presentation for the above image is given by:

```
<x,y,t|x^5, y^2, (x * y)^4,
 x * y * x^3 * y * x^2 * y * x^-2 * y * x, t^2, (t,y),
(t,(y*x^4)^(x^4)*y^(y*x)), (t,y^(y*x)),
(t,y^(y*x*y*x)), t*t^x = t^x*t>,
```

where $N = S_5 \cong$ `<x,y|x^5, y^2, (x * y)^4,`

`x * y * x^3 * y * x^2 * y * x^-2 * y * x>,`

and the action of x and y on the symmetric generators is given by

$x \sim (0\ 1\ 2\ 3\ 4)$

$y \sim (1\ 2)$.

Double Coset Enumeration

Before looking at any relations, it should be mentioned that G will not be symmetrically generated by $\{t_0, t_1, t_2, t_3, t_4\}$. As a matter of fact, the set of involutions as mentioned before will only generate a subgroup of G of index 120.

Expanding the relation $[(0\ 1)t_0]^4 = 1$ yields:

$\pi^4(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}(t_0) = 1$, where $\pi = (0\ 1) \in N$

$\Rightarrow (Id)t_1t_0t_1t_0 = 1$

$\Rightarrow \boxed{t_0t_1 = t_1t_0 \qquad (1)}$

$\Rightarrow t_it_j = t_jt_i$.

Therefore, $t_0t_1 = t_1t_0$.

In like manner, we have

$t_0t_2 = t_2t_0$	$t_1t_2 = t_2t_1$	$t_2t_3 = t_3t_2$
$t_0t_3 = t_3t_0$	$t_1t_3 = t_3t_1$	$t_2t_4 = t_4t_2$
$t_0t_4 = t_4t_0$	$t_1t_4 = t_4t_1$	$t_0t_1 = t_1t_0$.

So, the double coset [0 1] has $5 \cdot 4 = 20$ single cosets.

But since t_0t_1 has two names, then [0 1] only has 20/2 = 10 distinct single cosets.

Furthermore, applying t_0 to $t_0t_1 = t_1t_0$ yields:

$t_0t_1t_0 = t_1t_0t_0$

$= t_1$

but $t_1 \in [0]$, (since $t_1^{\pi} = t_0$, where $\pi = (0\ 1) \in N$).

Therefore, [0 1 0] = [0]. Since $N^{(0\ 1)}$ has orbits {0, 1} and {2, 3, 4} then [0 1 i] = [0] for all i in {0, 1}.

Now, from (1) we have

$t_0t_1 = t_1t_0$

$\Rightarrow t_0t_1t_2 = t_1\underline{t_0t_2}$

$= \underline{t_1t_2}t_0$ (since $t_0t_2 = t_2t_0$)

$= t_2\underline{t_1t_0}$ (since $t_1t_2 = t_2t_1$)

$= \underline{t_2t_0}t_1$ (since $t_1t_0 = t_0t_1$)

$= t_0t_2t_1$ (since $t_2t_0 = t_0t_2$).

Therefore,

$t_0t_1t_2 = t_1t_0t_2 = t_1t_2t_0 = t_2t_1t_0 = t_2t_0t_1 = t_0t_2t_1$.

So, the double coset [0 1 2] has $5 \cdot 4 \cdot 3 = 60$ single cosets. But since $t_0t_1t_2$ has six names, then [0 1 2] only has $60/6 = 10$ distinct single cosets, namely $Nt_0t_1t_2$, $Nt_0t_1t_3$, $Nt_0t_1t_4$, $Nt_0t_2t_3$, $Nt_0t_2t_4$, $Nt_0t_3t_4$, $Nt_1t_2t_3$, $Nt_1t_2t_4$, $Nt_1t_3t_4$, and $Nt_2t_3t_4$.

Furthermore, applying t_0 to $t_0t_1t_2 = t_1t_2t_0$ yields:

$t_0t_1t_2t_0 = t_1t_2t_0t_0$

$= t_1t_2$

but since $t_1t_2 \in$ [0 1] (since $(t_1t_2)^{\pi} = t_0t_1$,

where $\pi = (210) \in N$).

Therefore, [0 1 2 0] = [0 1]. Since $N^{(0\ 1\ 2)}$ has orbits {0, 1, 2} and {3, 4} then [0 1 2 j] = [0 1] for all j in {0, 1, 2}.

Now, applying t_3 to $t_0t_1t_2 = t_1t_2t_0$ yields:

$t_0t_1t_2t_3 = t_1t_2t_0t_3$

$= t_2t_1t_0t_3$

$= t_2t_0t_1t_3$

$= t_2t_0t_3t_1$

$= t_2t_3t_0t_1$

$= t_3t_2t_0t_1$

$= t_3t_0t_2t_1$

$= t_3t_0t_1t_2$

$= t_3t_1t_0t_2$

$= t_1t_3t_0t_2$

$= t_1t_0t_3t_2$

$= t_1t_0t_2t_3$

$= t_0t_2t_1t_3$

$= t_0t_2t_3t_1$

$= t_0t_3t_2t_1$

$= t_0t_3t_1t_2$

$= t_0t_1t_3t_2$.

Now,

$t_0t_1t_2t_3 = t_3t_2t_0t_1$

$= t_3t_2t_1t_0$

$= t_3t_1t_2t_0$

$= t_1t_3t_2t_0$

$= t_1t_2t_3t_0$

$= t_2t_1t_3t_0$

$= t_2t_3t_1t_0$.

Therefore,

$t_0t_1t_2t_3 = t_1t_2t_0t_3 = t_2t_1t_0t_3 = t_2t_0t_1t_3 = t_2t_0t_3t_1 = t_2t_3t_0t_1 =$

$t_3t_2t_0t_1 = t_3t_0t_2t_1 = t_3t_0t_1t_2 = t_3t_1t_0t_2 = t_1t_3t_0t_2 = t_1t_0t_3t_2 =$

$t_1t_0t_2t_3 = t_0t_2t_1t_3 = t_0t_2t_3t_1 = t_0t_3t_2t_1 = t_0t_3t_1t_2 = t_0t_1t_3t_2 =$

$t_3t_2t_1t_0 = t_3t_1t_2t_0 = t_1t_3t_2t_0 = t_1t_2t_3t_0 = t_2t_1t_3t_0 = t_2t_3t_1t_0.$

Therefore, the double coset [0 1 2 3] has $5 \cdot 4 \cdot 3 \cdot 2 =$ 120 single cosets. But since $t_0t_1t_2t_3$ has twenty four names, then [0 1 2 3] only has 120/24 = 5 distinct single cosets, namely $Nt_0t_1t_2t_3$, $Nt_0t_1t_2t_4$, $Nt_0t_1t_3t_4$, $Nt_0t_2t_3t_4$, and $Nt_1t_2t_3t_4$.

Now, applying t_0 to $t_0t_1t_2t_3 = t_2t_3t_1t_0$ yields:

$t_0t_1t_2t_3t_0 = t_2t_3t_1t_0t_0$

$= t_2t_3t_1$

but $t_2t_3t_1 \in$ [0 1 2] (since $(t_2t_3t_1)^{\pi} = t_0t_1t_2$,

where $\pi = (3\ 1\ 2\ 0) \in N$).

Therefore, [0 1 2 3 0] = [0 1 2]. Since $N^{(0\ 1\ 2\ 3)}$ has orbits {4} and {0, 1, 2, 3} then [0 1 2 3 k] = [0 1 2] for all k in {0, 1, 2, 3}.

Now, applying t_4 to $t_0t_1t_2t_3 = t_1t_2t_0t_3$ yields:

$\Rightarrow t_0t_1t_2t_3t_4 = t_1t_2t_0t_3t_4$

$\Rightarrow t_0t_1t_2t_3t_4 = t_1t_2t_0t_4t_3.$

Using the above relation to calculate the order of the stabilizing group $N^{(0\ 1\ 2\ 3\ 4)}$ (see Appendix C(2)) we see that its order equals 120. Therefore, the number of distinct

single cosets in the double coset [0 1 2 3 4] turns out to be $\frac{|N|}{|N^{(0\ 1\ 2\ 3\ 4)}|} = \frac{120}{120} = 1$, which implies that all the single cosets in [0 1 2 3 4] are all equal to each other.

Now, applying t_3 to $t_0t_1t_2t_3t_4 = t_4t_1t_2t_0t_3$ yields:

$$t_0t_1t_2t_3t_4t_3 = t_4t_1t_2t_0t_3t_3$$
$$= t_4t_1t_2t_0$$

but $t_4t_1t_2t_0 \in$ [0 1 2 3] (since $(t_4t_1t_2t_0)^{\pi} = t_0t_1t_2t_3$, where $\pi = (4\ 0\ 3) \in N$).

Therefore, [0 1 2 3 4 3] = [0 1 2 3]. Similarly, [0 1 2 3 4 m] = [0 1 2 3 4] for all m in {0, 1, 2, 3, 4}, which implies that $N^{(0\ 1\ 2\ 3\ 4)}$ is transitive. Therefore, we must have found all the double cosets as illustrated in Figure 8.3.

The labeling used for the single cosets of N in G is given by:

Label	Single Coset
32.	[N]
1.	[1]
2.	[2]
3.	[3]
4.	[4]
5.	[0]
6.	[0 1]
7.	[0 2]
8.	[0 3]
9.	[0 4]
10.	[1 2]

11. [1 3]
12. [1 4]
13. [2 3]
14. [2 4]
15. [3 4]
16. [0 1 2]
17. [0 1 3]
18. [0 1 4]
19. [0 2 3]
20. [0 2 4]
21. [0 3 4]
22. [1 2 3]
23. [1 2 4]
24. [1 3 4]
25. [2 3 4]
26. [0 1 2 3]
27. [0 1 2 4]
28. [0 1 3 4]
29. [0 2 3 4]
30. [1 2 3 4]
31. [0 1 2 3 4]

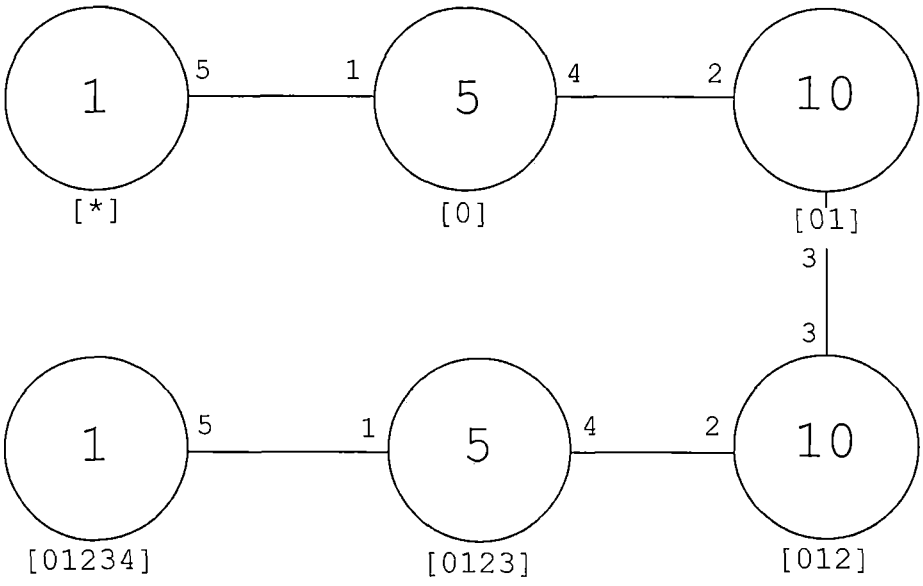

Figure 8.3 Cayley Graph of 2^4 : (S_5 x 2) over S_5

The action of the symmetric generators on the cosets of N in G is given by:

t_0 = (32,5)(1,6)(2,7)(3,8)(4,9)(10,16)(11,17)(12,18)(13,19)
(14,20)(15,21)(22,26)(23,27)(24,28)(25,29)(30,31)

t_1 = (32,1)(5,6)(2,10)(3,11)(4,12)(7,16)(8,17)(9,18)(13,22)
(14,23)(15,24)(19,26)(20,27)(21,28)(25,30)(29,31)

t_2 = (32,2)(5,7)(1,10)(3,13)(4,14)(6,16)(8,19)(9,20)(11,22)
(12,23)(15,25)(17,26)(18,27)(21,29)(24,30)(28,31)

t_3 = (32,3)(5,8)(1,11)(2,13)(4,15)(6,17)(7,19)(9,21)(10,22)
(12,24)(14,25)(16,26)(18,28)(20,29)(23,30)(27,31)

t_4 = (32,4)(5,9)(1,12)(2,14)(3,15)(6,18)(7,20)(8,21)(10,23)
(11,24)(13,25)(16,27)(17,28)(19,29)(22,30)(26,31)

Group $2^4 : S_5$

Factoring the progenitor $2^{*5} : S_5$ by the following relations

$$[(0\ 1)t_0]^4 \text{ and } [(0\ 1\ 2\ 3\ 4)t_0]^5$$

yields the finite homomorphic image:

$$G \cong 2^4 : S_5 \cong \frac{2^{*5} : S_5}{[(0\ 1)t_0]^4,\ [(0\ 1\ 2\ 3\ 4)t_0]^5}.$$

A symmetric presentation for the above image is given by:

<x,y,t|x^5, y^2, (x * y)^4,

x * y * x^3 * y * x^2 * y * x^-2 * y * x, t^2, (t,y),

(t,(y*x^4)^(x^4)*y^(y*x)), (t,y^(y*x)),

(t,y^(y*x*y*x)), t*t^x= t^(x^4)*t^(x^3)*t^(x^2)>,

where N = S_5 ≅ <x,y,t|x^5, y^2, (x * y)^4,

x * y * x^3 * y * x^2 * y * x^-2 * y * x>,

and the action of x and y on the symmetric generators is given by

$x \sim (0\ 1\ 2\ 3\ 4)$

$y \sim (1\ 2)$.

Double Coset Enumeration:

Before looking at any relations, it should be mentioned that G will not be symmetrically generated by $\{t_0, t_1, t_2, t_3, t_4\}$. As a matter of fact, the set of involutions as mentioned before will only generate a subgroup of G of index 32.

Expanding the relations $[(0\ 1)t_0]^4 = 1$ yields:

$(\pi t_0)^4 = 1$, where $\pi = (0\ 1) \in N$

$\Rightarrow \pi^4(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi^1}(t_0)^{\pi^2} = 1$

$\Rightarrow (Id)t_1t_0 = t_0t_1$

$\Rightarrow \boxed{t_0t_1 = t_1t_0 \qquad (1)}$

$\Rightarrow t_it_j = t_jt_i$ for all i, j in {0, 1, 2, 3, 4}.

So, the double coset [0 1] has $5 \cdot 4 = 20$ single cosets. But since t_0t_1 has two names, then [0 1] only has $20/2 = 10$ distinct single cosets, namely Nt_0t_1, Nt_0t_2, Nt_0t_3, Nt_0t_4, Nt_1t_2, Nt_1t_3, Nt_1t_4, Nt_2t_3, Nt_2t_4, and Nt_3t_4.

Now, applying t_0 to (1) we have

$t_0t_1 = t_1t_0$

$\Rightarrow t_0t_1t_0 = t_1t_0t_0$

$= t_1,$

but $t_1 \in [0]$, (since $t_1^{\pi} = t_0$, where $\pi = (0\ 1) \in N$).

Therefore, $[0\ 1\ 0] = [0]$. Since $N^{(0\ 1)}$ has orbits $\{0, 1\}$ and $\{2, 3, 4\}$ then $[0\ 1\ i] = [0]$ for all i in $\{0, 1\}$.

Now, expanding the relation $[(0\ 1\ 2\ 3\ 4)t_0]^5 = 1$ yields:

$\pi^5(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi^1}(t_0)^{\pi^2} = 1$, where $\pi = (0\ 1\ 2\ 3\ 4) \in N$

$\Rightarrow (Id)t_4t_3t_2t_1t_0 = 1$

$\Rightarrow t_0t_1t_2 = t_4t_3$

but $t_4t_3 \in [0\ 1]$, (since $(t_4t_3)^{\pi} = t_0t_1$,

where $\pi = (0\ 4)(1\ 3) \in N$).

Therefore, $[0\ 1\ 2] = [0\ 1]$, which implies that $[0\ 1\ j] = [0\ 1]$ for all j in $\{2, 3, 4\}$. So, applying either t_2, t_3, or t_4 to $t_0t_1 \in [0\ 1]$ will loop back to the same double coset.

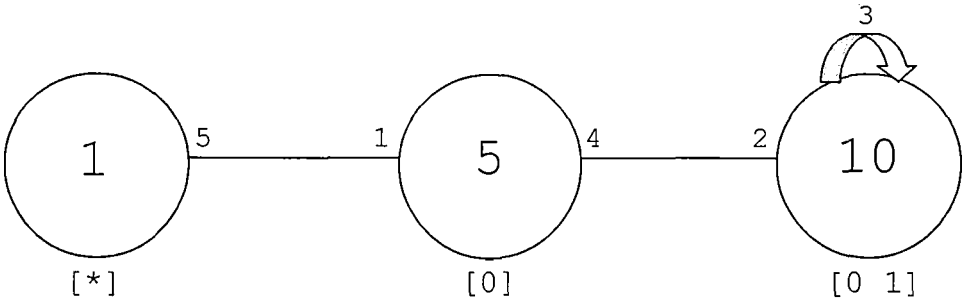

Figure 8.4 Cayley Graph of $2^4 : S_5$ over S_5

Therefore, we must have found all the double cosets as illustrated in Figure 8.4.

The labeling used for the single cosets of N in G is given by:

Label	Single Coset
16.	[N]
1.	[1]
2.	[2]
3.	[3]
4.	[4]
5.	[0]
6.	[0 1]
7.	[0 2]
8.	[0 3]
9.	[0 4]
10.	[1 2]
11.	[1 3]
12.	[1 4]
13.	[2 3]
14.	[2 4]
15.	[3 4]

The action of the symmetric generators on the cosets of N in G is given by:

t_0 = (16,5)(1,6)(2,7)(3,8)(4,9)(10,15)(11,14)(12,13)

t_1 = (16,1)(5,6)(2,10)(3,11)(4,12)(7,15)(8,14)(9,13)

t_2 = (16,2)(5,7)(1,10)(3,13)(4,14)(6,15)(8,12)(9,11)

t_3 = (16,3)(5,8)(1,11)(2,13)(4,15)(6,14)(7,12)(9,10)

t_4 = (16,4)(5,9)(1,12)(2,14)(3,15)(6,13)(7,11)(8,10).

CHAPTER NINE

SYMMETRIC REPRESENTAION OF $P\Sigma L_2(49)$ ELEMENTS

Introduction

In this chapter we will show that the group $P\Sigma L_2(49)$ is generated by 5 involutions which are permuted, by conjugation within $P\Sigma L_2(49)$, by a subgroup isomorphic to S_5, and that each element of $P\Sigma L_2(49)$ can be written (not necessarily uniquely) as a permutation of S_5 on 5 letters followed by a word of length less than or equal to seven in these generators.

Factoring the progenitor $2^{*5} : S_5$ by the following relations

$$[(0\ 1\ 2)(3\ 4)t_0]^7,\ [(0\ 1\ 2\ 3\ 4)t_0]^8,\ \text{and}\ [(0\ 1)t_0]^8$$

yields the finite homomorphic image:

$$G \cong \frac{2^{*5} : S_5}{[(0\ 1\ 2)(3\ 4)t_0]^7,\ \ [(0\ 1\ 2\ 3\ 4)t_0]^8,\ \ [(0\ 1)t_0]^8},$$

where the index of S_5 in G equals 980. $G \cong P\Sigma L_2(49)$, the projective sigma linear group.

A symmetric presentation for the above image is given by:

```
<x,y,t|x^5, y^2, (x * y)^4, x * y * x^3 * y * x^2 * y
* x^-2 * y * x, t^2, (t,y), (t, y^(y*x*y*x)), (t,
((y*x^4)^(x^4)*y^(y*x))^(y^(x*y))),
(((y*x^2)^(y^(x*y)))^(-1)*t)^7>,
```

where $N = S_5 \cong$ <x,y,t|x^5, y^2, (x * y)^4, x * y * x^3 * y * x^2 * y * x^-2 * y * x>,

and the action of x, y on the symmetric generators is given by

$x \sim (1\ 2\ 3\ 4\ 0)$

$y \sim (1\ 2)$.

Double Coset Enumeration

The double coset [0] has 5 distinct single cosets, namely Nt_0, Nt_1, Nt_2, Nt_3, and Nt_4. Meanwhile, the double cosets [0 1] and [0 1 0] each contain 20 single cosets where $N^{(0\ 1)}$ and $N^{(0\ 1\ 0)}$ both have orbits {0}, {1}, and {2, 3, 4}.

Now, expanding the relation $[(0\ 1)t_0]^8 = 1$ yields:

$(\pi t_0)^8 = 1$, where $\pi = (0\ 1) \in N$

$$\Rightarrow \pi^8(t_0)^{\pi^7}(t_0)^{\pi^6}(t_0)^{\pi^5}(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}t_0 = 1$$

$$\Rightarrow t_1t_0t_1t_0t_1t_0t_1t_0 = 1$$

$$\Rightarrow \boxed{t_0t_1t_0t_1 = t_1t_0t_1t_0. \qquad (1)}$$

The double coset [0 1 0 1] contains $5 \cdot 4 \cdot 1 \cdot 1 = 20$ single cosets, but since $t_0t_1t_0t_1$ has two names, then the number of distinct single cosets in [0 1 0 1] $= \frac{20}{2} = 10$.

Applying t_0 to (1) yields:

$t_0t_1t_0t_1t_0 = t_1t_0t_1t_0t_0$

$= t_1t_0t_1t_0t_0$

$= t_1t_0t_1,$

but $t_1t_0t_1 \in [0\ 1\ 0]$ (since $(t_1t_0t_1)^{\pi} = t_0t_1t_0,$

where $\pi = (0\ 1) \in N$).

Therefore, $[0\ 1\ 0\ 1\ 0] = [0\ 1\ 0]$.

Since $N^{(0\ 1\ 0\ 1)}$ has orbits {0, 1} and {2, 3, 4}, then

$[0\ 1\ 0\ 1\ i] = [0\ 1\ 0]$ for all i in {0, 1}.

Applying t_2 to (1) yields:

$t_0t_1t_0t_1t_2 = t_1t_0t_1t_0t_2$.

The double coset $[0\ 1\ 0\ 1\ 2]$ contains $5 \cdot 4 \cdot 1 \cdot 1 \cdot 3 =$ 60 single cosets, but since $t_0t_1t_0t_1t_2$ has two names, then the number of distinct single cosets in $[0\ 1\ 0\ 1\ 2] = \frac{60}{2} =$ 30.

Now, $t_0t_1t_0t_1t_2t_0$ can be written as

$t_0t_1t_0t_1t_2t_0 = t_0t_1(1\ 2\ 0)(3\ 4)t_0t_2t_1$

$= (1\ 2\ 0)(3\ 4)t_1t_2t_0t_2t_1,$

but $t_1t_2t_0t_2t_1 \in [0\ 1\ 2\ 1\ 0]$ (since $(t_1t_2t_0t_2t_1)^{\pi} = t_0t_1t_2t_1t_0,$

where $\pi = (0\ 2\ 1) \in N$).

Therefore, $[0\ 1\ 0\ 1\ 2\ 0] = [0\ 1\ 2\ 1\ 0]$.

Since $N^{(0\ 1\ 0\ 1\ 2)}$ has orbits {2}, {0, 1}, and {3, 4}, then

$[0\ 1\ 0\ 1\ 2\ j] = [0\ 1\ 2\ 1\ 0]$ for all j in {0, 1}.

Similarly, $[0\ 1\ 0\ 2\ h] = [0\ 1\ 2\ 1]$ for all h in {3, 4}.

The double coset [0 1 0 2] has 60 single cosets.

Now, $t_0t_1t_0t_2t_1$ can be written as

$t_0t_1t_0t_2t_1 = t_0(0\ 2\ 1)(3\ 4)t_1t_2t_0$

$= (0\ 2\ 1)(3\ 4)t_2t_1t_2t_0,$

but $t_2t_1t_2t_0 \in [0\ 1\ 0\ 2]$ (since $(t_2t_1t_2t_0)^\pi = t_0t_1t_0t_2$,

where $\pi = (0\ 2) \in N$).

Therefore, [0 1 0 2 1] = [0 1 0 2].

Similarly, since $N^{(0\ 1\ 0\ 2)}$ has orbits, {0}, {1}, {2}, and {3, 4} then [0 1 0 2 k] = [0 1 2 3 0] for all k in {3, 4}.

The double coset [0 1 0 2 0] contains 60 single cosets.

Now, $t_0t_1t_0t_2t_0t_1$ can be written as

$t_0t_1t_0t_2t_0t_1 = t_0t_1(0\ 1\ 2)(3\ 4)t_1t_2t_1t_0t_2$

$= (0\ 1\ 2)(3\ 4)t_1t_2t_1t_2t_1t_0t_2$

$= (0\ 1\ 2)(3\ 4)t_2t_1t_2t_1t_1t_0t_2$

$= (0\ 1\ 2)(3\ 4)t_2t_1t_2t_0t_2,$

but $t_2t_1t_2t_0t_2 \in [0\ 1\ 0\ 2\ 0]$ (since $(t_2t_1t_2t_0t_2)^\pi = t_0t_1t_0t_2t_0$,

where $\pi = (0\ 2) \in N$).

Therefore, [0 1 0 2 0 1] = [0 1 0 2 0].

Now, $t_0t_1t_0t_2t_0t_2$ can be written as

$t_0t_1t_0t_2t_0t_2 = t_0t_1t_2t_0t_2t_0$

$= (0\ 1\ 2)(3\ 4)t_0t_2t_1t_2t_0,$

but $t_0t_2t_1t_2t_0 \in [0\ 1\ 2\ 1\ 0]$ (since $(t_0t_2t_1t_2t_0)^\pi = t_0t_1t_2t_1t_0$,

where $\pi = (1\ 2) \in N$).

Therefore, [0 1 0 2 0 2] = [0 1 2 1 0].

Similarly, since $N^{(0\ 1\ 0\ 2\ 0)}$ has orbits, {0}, {1}, {2}, and {3, 4} then [0 1 0 2 0 m] = [0 1 2 1 3 1] for all m in {3, 4}.

The double coset [0 1 2] contains 60 single cosets where $N^{(0\ 1\ 2)}$ has orbits {0}, {1}, {2}, and {3, 4}.

Now, expanding the relation $[(0\ 1\ 2)(3\ 4)t_0]^7$ yields:

$(\pi t_0)^7 = 1$, where $\pi = (0\ 1\ 2)(3\ 4) \in N$

$\Rightarrow \pi^7(t_0)^{\pi^6}(t_0)^{\pi^5}(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}t_0 = 1$

$\Rightarrow (1\ 2\ 0)(3\ 4)t_0t_2t_1t_0t_2t_1t_0 = 1$

$\Rightarrow \boxed{t_0t_1t_2t_0 = (1\ 2\ 0)(3\ 4)t_0t_2t_1,\ (2)}$

but $t_0t_2t_1 \in [0\ 1\ 2]$ (since $(t_0t_2t_1)^{\pi} = t_0t_1t_2$, where $\pi = (1\ 2) \in N$).

Therefore, [0 1 2 0] = [0 1 2].

The double cosets [0 1 2 1] and [0 1 2 1 0] each contain 60 single cosets where $N^{(0\ 1\ 2\ 1)}$ and $N^{(0\ 1\ 2\ 1\ 0)}$ both have orbits {0}, {1}, {2}, {3, 4}.

Furthermore, it can be easily verified that [0 1 2 1 2] = [0 1 0 1 2] and [0 1 2 1 0 1] = [0 1 0 2 0].

Now, $t_0t_1t_2t_1t_0t_2$ can be written as

$t_0t_1t_2t_1t_0t_2 = t_0t_1(1\ 0\ 2)(3\ 4)t_2t_0t_1$

$= (1\ 0\ 2)(3\ 4)t_2t_0t_2t_0t_1,$

but $t_2t_0t_2t_0t_1 \in [0\ 1\ 0\ 1\ 2]$ (since $(t_2t_0t_2t_0t_1)^{\pi} = t_0t_1t_0t_1t_2$, where $\pi = (2\ 0\ 1) \in N$).

Therefore, $[0\ 1\ 2\ 1\ 0\ 2] = [0\ 1\ 0\ 1\ 2]$.

The double coset $[0\ 1\ 2\ 1\ 0\ 3]$ 120 single cosets.

Now, $t_0t_1t_2t_1t_0t_3t_4$ can be written as

$$t_0t_1t_2t_1t_0t_3t_4 = t_0t_1t_2(3\ 1\ 4\ 0\ 2)t_3t_0t_1t_2$$
$$= (3\ 1\ 4\ 0\ 2)t_2t_4t_3t_3t_0t_1t_2$$
$$= (3\ 1\ 4\ 0\ 2)t_2t_4t_0t_1t_2$$

but $t_2t_4t_0t_1t_2 \in [0\ 1\ 2\ 3\ 0]$ (since $(t_2t_4t_0t_1t_2)^{\pi} = t_0t_1t_2t_3t_0$, where $\pi = (0\ 2)(1\ 3\ 4) \in N$).

Therefore, $[0\ 1\ 2\ 1\ 0\ 3\ 4] = [0\ 1\ 2\ 3\ 0]$.

Similarly, $[0\ 1\ 2\ 1\ 0\ 3\ 1] = [0\ 1\ 2\ 1\ 0\ 3]$ and $[0\ 1\ 2\ 1\ 0\ 3\ 2] = [0\ 1\ 2\ 1\ 3\ 1]$.

The double coset $[0\ 1\ 2\ 1\ 0\ 3\ 0]$ contains 120 single cosets. However, using MAGMA we can see that $t_0t_1t_2t_1t_0t_3t_0$ has four names (see Appendix D(1)), namely $Nt_0t_1t_2t_1t_0t_3t_0 = Nt_3t_0t_2t_0t_3t_4t_3 = Nt_1t_4t_2t_4t_1t_0t_1 = t_4t_3t_2t_3t_4t_1t_4$. Therefore, the number of distinct single cosets in $[0\ 1\ 2\ 1\ 0\ 3\ 0]$ is given by $\frac{120}{4} = 30$.

Now, applying t_3 to $Nt_0t_1t_2t_1t_0t_3t_0 = Nt_3t_0t_2t_0t_3t_4t_3$ yields:

$$Nt_0t_1t_2t_1t_0t_3t_0t_3 = Nt_3t_0t_2t_0t_3t_4t_3t_3$$
$$= Nt_3t_0t_2t_0t_3t_4,$$

but $t_3t_0t_2t_0t_3t_4 \in [0\ 1\ 2\ 1\ 0\ 3]$

(since $(Nt_3t_0t_2t_0t_3t_4t_3)^{\pi} = Nt_0t_1t_2t_1t_0t_3$, where $\pi = (1\ 4\ 3\ 0) \in N$).

Therefore [0 1 2 1 0 3 0 3] = [0 1 2 1 0 3].

However, since $N^{(0\ 1\ 2\ 1\ 0\ 3\ 0)}$ has orbits {2} and {0, 1, 3, 4} then [0 1 2 1 0 3 0 p] = [0 1 2 1 0 3] for all p in {0, 1, 3, 4}.

Similarly, [0 1 2 1 0 3 0 2] = [0 1 2 1 0 3 0].

Now, expanding the relation $[(0\ 1\ 2\ 3\ 4)t_0]^8 = 1$ yields:

$(\pi t_0)^8 = 1$, where $\pi = (0\ 1\ 2\ 3\ 4) \in N$

$\Rightarrow \pi^8(t_0)^{\pi^7}(t_0)^{\pi^6}(t_0)^{\pi^5}(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}t_0 = 1$

$\Rightarrow (0\ 3\ 1\ 4\ 2)t_2t_1t_0t_4t_3t_2t_1t_0 = 1$

$\Rightarrow \boxed{t_0t_1t_2t_3 = (0\ 3\ 1\ 4\ 2)t_2t_1t_0t_4.} \quad (3)$

The double coset [0 1 2 3] contains $5 \cdot 4 \cdot 3 \cdot 2 = 120$ single cosets. However, since $t_0t_1t_2t_3$ has two names, then the number of distinct single cosets in [0 1 2 3] turns out to be $\frac{120}{2} = 60$.

Now, applying t_4 to (3) yields:

$t_0t_1t_2t_3t_4 = (0\ 3\ 1\ 4\ 2)t_2t_1t_0t_4t_4$

$= (0\ 3\ 1\ 4\ 2)t_2t_1t_0,$

but $t_2t_1t_0 \in [0\ 1\ 2]$ (since $(t_2t_1t_0)^{\pi} = t_0t_1t_2$, where $\pi = (0\ 2) \in N$).

Therefore, [0 1 2 3 4] = [0 1 2], but since $N^{(0\ 1\ 2\ 3)}$ has orbits {1}, {0, 2}, and {3, 4} then [0 1 2 3 q] = [0 1 2] for all q in {3, 4}.

Now, $t_0t_1t_2t_3t_1$ can be written as

$$t_0t_1t_2t_3t_1 = (0\ 3\ 1\ 4\ 2)t_2t_1t_0t_4t_1$$
$$= (0\ 3\ 1\ 4\ 2)t_2(0\ 4\ 1)(3\ 2)t_1t_4t_0$$
$$= (2\ 4\ 3\ 0)t_3t_1t_4t_0,$$

but $t_3t_1t_4t_0 \in$ [0 1 2 3] (since $(t_3t_1t_4t_0)^{\pi} = t_0t_1t_2t_3$, where $\pi = (2\ 4)(3\ 0) \in N$).

Therefore, [0 1 2 3 1] = [0 1 2 3].

The double coset [0 1 2 3 0] has 120 cosets.

Now, $t_0t_1t_2t_3t_0t_2$ can be written as

$$t_0t_1t_2t_3t_0t_2 = (3\ 0\ 2)(1\ 4)t_2t_4t_2t_0t_3$$
$$= (3\ 0\ 2)(1\ 4)t_2(4\ 3\ 2\ 1\ 02)t_0t_2t_4t_1$$
$$= (1\ 3\ 4\ 0)t_1t_0t_2t_4t_1,$$

but $t_1t_0t_2t_4t_1 \in$ [0 1 2 3 0] (since $(t_1t_0t_2t_4t_1)^{\pi} = t_0t_1t_2t_3t_0$, where $\pi = (0\ 1)(3\ 4) \in N$).

Therefore, [0 1 2 3 0 2] = [0 1 2 3 0].

Now, $t_0t_1t_2t_3t_0t_4$ can be written as

$$t_0t_1t_2t_3t_0t_4 = t_0t_1(2\ 4\ 3\ 1\ 0)t_0t_3t_2t_1$$
$$= (2\ 4\ 3\ 1\ 0)t_2t_0t_0t_3t_2t_1,$$
$$= (2\ 4\ 3\ 1\ 0)t_2t_3t_2t_1,$$

but $t_2t_3t_2t_1 \in [0\ 1\ 0\ 2]$ (since $(t_2t_3t_2t_1)^{\pi} = t_0t_1t_0t_2$,

where $\pi = (3\ 1\ 2\ 0) \in N$).

Therefore, [0 1 2 3 0 4] = [0 1 0 2].

Similarly, [0 1 2 3 0 3] = [0 1 2 1 3] and [0 1 2 3 0 1] = [0 1 2 1 0 3].

The double coset [0 1 2 1 3] has 120 single cosets.

Now, $t_0t_1t_2t_1t_3t_2$ can be written as

$t_0t_1t_2t_1t_3t_2 = t_0t_1(1\ 3\ 2)(0\ 4)t_2t_3t_1$

$= (1\ 3\ 2)(0\ 4)t_4t_3t_2t_3t_1$,

but $t_4t_3t_2t_3t_1 \in [0\ 1\ 2\ 1\ 3]$ (since $(t_4t_3t_2t_3t_1)^{\pi} = t_0t_1t_2t_1t_3$,

where $\pi = (1\ 3)(0\ 4) \in N$).

Therefore, [0 1 2 1 3 2] = [0 1 2 1 3].

Similarly, [0 1 2 1 3 4] = [0 1 2 1 3].

Now, $t_0t_1t_2t_1t_3t_0$ can be written as

$t_0t_1t_2t_1t_3t_0 = t_0t_1(2\ 0\ 1\ 4\ 3)t_3t_1t_2t_4$

$= (2\ 0\ 1\ 4\ 3)t_1t_4t_3t_1t_2t_4$

$= (1\ 3\ 0\ 4)t_1t_3t_4t_2t_4$

$= (1\ 0)(2\ 3\ 4)t_4t_3t_1t_0t_4$

but $t_4t_3t_1t_0t_4 \in [0\ 1\ 2\ 3\ 0]$ (since $(t_4t_3t_2t_1t_0t_4)^{\pi} = t_0t_1t_2t_3t_0$,

where $\pi = (1\ 2\ 4\ 0\ 3) \in N$).

Therefore, [0 1 2 1 3 0] = [0 1 2 3 0].

The double coset [0 1 2 1 3 1] has 120 single cosets where $N^{(0\ 1\ 2\ 1\ 3\ 1)}$ has orbits {0}, {1}, {2}, {3}, and {4}.

Furthermore, [0 1 2 1 3 1 3] = [0 1 2 1 0 3], [0 1 2 1 3 1 2] = [0 1 2 1 3 1], and [0 1 2 1 3 1 4] = [0 1 0 2 0].

The double coset [0 1 2 1 3 1 0] contains 120 single cosets. However, using MAGMA we can see that $t_0t_1t_2t_1t_3t_1t_0$ has five names (see Appendix D(2)), namely $Nt_0t_1t_2t_1t_3t_1t_0 = Nt_3t_2t_0t_2t_4t_2t_3 = Nt_4t_0t_3t_0t_1t_0t_4 = Nt_1t_3t_4t_3t_2t_3t_1 = Nt_2t_4t_1t_4t_0t_4t_2$. Therefore, the number of distinct single cosets in [0 1 2 1 3 1 0] $= \frac{120}{5} = 24$.

Now applying t_3 to $Nt_0t_1t_2t_1t_3t_1t_0 = Nt_3t_2t_0t_2t_4t_2t_3$ yields:

$$Nt_0t_1t_2t_1t_3t_1t_0t_3 = Nt_3t_2t_0t_2t_4t_2t_3t_3$$
$$= Nt_3t_2t_0t_2t_4t_2,$$

but $t_3t_2t_0t_2t_4t_2 \in$ [0 1 2 1 3 1]

(since $(t_3t_2t_0t_2t_4t_2)^{\pi} = t_0t_1t_2t_1t_3t_1$, where $\pi = (1, 4, 3, 0, 2) \in N$).

Therefore, [0 1 2 1 3 1 0 3] = [0 1 2 1 3 1].

Similarly, since $N^{(0\ 1\ 2\ 1\ 3\ 1\ 0)}$ is transitive on $\{t_0, t_1, t_2, t_3, t_4\}$, i.e. has orbit {0, 1, 2, 3, 4}, then [0 1 2 1 3 1 0 r] = [0 1 2 1 3] for all r in {0, 1, 2, 3, 4}.

The set of all double cosets [w] = NwN, coset stabilizing subgroups $N^{(w)}$, and the number of single cosets are exhibited in Table 9.1. The table shows that the Cayley Graph of G over N has the form as displayed in Figure 9.1.

Table 9.1 The Double Cosets [w] = NwN in G, Where N = S_5

NwN	Reason	Number of single cosets
[*]	N is transitive	1
[0]	$N^{(0)}$ = <*e*> has orbits {0}, {1, 2, 3, 4}	5
[00] = [*]		
[01]	$N^{(0\ 1)}$ = <(24), (234)> has orbits {0}, {1}, {2, 3, 4}	20
[010]	$N^{(0\ 1\ 0)}$ = <(24), (234)> has orbits {0}, {1}, {2, 3, 4}	20
[0101]	$N^{(0\ 1\ 0\ 1)}$ = <(234), (24), (01), (01)(243), (01)(23), (01)(234), (01)(34), (01)(24)> has orbits {0, 1}, {2, 3, 4}, 0101 = 1010 and for all i in {0, 1} [0101i]=[010]	20
[01012]	$N^{(0\ 1\ 0\ 1\ 2)}$ = <(01), (01)(34)> has orbits {2}, {0, 1}, {3, 4} and 01012 = 10102, also 010120 = 01(120)(34)021 = (120)(34)12021 and $(12021)^{\pi}$ = 01210 where π = (021) $\in$ N [010120]=[01210]	30
	010123 = (01)(34)2434 and $(2434)^{\pi}$ = 0121 where π = (14)(032) $\in$ N [010123]=[0121]	
[0102]	$N^{(0\ 1\ 0\ 2)}$ = <(34)> has orbits {0}, {1}, {2}, {3, 4}, 01021 = 0(021)(34)120 = (021)(34)2120 and $(2120)^{\pi}$ = 0102 where π = (02) $\in$ N [01021]=[0102]	60
	01023 = (04213)42014 and $(42014)^{\pi}$ = 01230 where π = (02134) $\in$ N [01023]=[01230]	

[01020] $N^{(0\ 1\ 0\ 2\ 0)}$ = <(34)> has orbits {0}, {1}, {2}, {3, 4}, 010201 = 01(012)(34)12102 = (012)(34)1212102 = (012)(34)21202 and $(21202)^{\pi}$ = 01020 where π = (02) $\in$ N — 60

[010201]=[01020]

010202 = 012020 = (120)(34)02120 and $(02120)^{\pi}$ = 012120 where π = (12) $\in$ N

[010202]=[01210]

010203 = (034)042414 and $(042414)^{\pi}$ = 012131 where π = (134) $\in$ N

[010203]=[12131]

[012] $N^{(0\ 1\ 2)}$ = <(34)> has orbits {0}, {1}, {2}, {3, 4}, 0120 = (120)(34)021 and $(021)^{\pi}$ = 012 where π = (12) $\in$ N — 60

[0120]=[012]

[0121] $N^{(0\ 1\ 2\ 1)}$ = <(34)> has orbits {0}, {1}, {2}, {3, 4}, 01212 = (12)(34)34340 and $(34340)^{\pi}$ = 01012 where π = (023)(14) $\in$ N — 60

[01212]=[01012]

[01210] $N^{(0\ 1\ 2\ 1\ 0)}$ <(34)> has orbits {0}, {1}, {2}, {3, 4}, 012102 = 01(102)(34)201 = (102)(34)20201 and $(20201)^{\pi}$ = 01012 where π = (012) $\in$ N — 60

[012102]=[01012]

012101 = (012)(34)02010 and $(02012)^{\pi}$ = 01020 where π = (12) $\in$ N

[012101]=[01020]

[012103] $N^{(0\ 1\ 2\ 1\ 0\ 3)}$ = <*e*> has orbits {0}, {1}, {2}, {3}, {4}, 0121034 = 012(31402)3012 = (31402)24012 and $(24012)^{\pi}$ = 01230 where π = (02)(134) $\in$ N — 120

[0121034]=[01230]

0121031 = (0432)413142 and $(413142)^{\pi}$ = 012103 where π = (04)(23) $\in$ N

[0121031]=[012103]

0121032 = (1432)143424 and $(143424)^{\pi}$ =
012131 where π = (23)(041) $\in$ N
[0121032]=[012131]

[0121030] $N^{(0\ 1\ 2\ 1\ 0\ 3\ 0)}$ = <(1034)> has orbits {2} 30
{1, 3, 4, 0}, 0121030 ~ 3020343 ~ 1424101
~ 4323414 and for all j in {0, 1, 3, 4}
[0121030j]=[012103]

01210302 = (01)(432)1020141 and $(1020141)^{\pi}$
= 0121030 where π = (01)(34) $\in$ N
[01210302]=[0121030]

[0123] $N^{(0\ 1\ 2\ 3)}$ = <(20)(34)> has orbits {1}, {0, 2}, 60
{3, 4}, 0123 = (03142)2104 and
01231 = (03142)21041 = (03142)2(041)(32)140
= (2430)3140 and $(3140)^{\pi}$ = 0123, where
π = (24)(03) $\in$ N
[01231]=[0123]

[01230] $N^{(0\ 1\ 2\ 3\ 0)}$ = <*e*> has orbits {0}, {1}, {2}, 120
{3}, {4}, 012302 = (302)(14)24203 =
(302)(14)2(43210)0241 = (1340)10241 and
$(10241)^{\pi}$ = 01230 where π = (01)(34) $\in$ N
[012302]=[01230]

01234 = 01(24310)0321 = (24310)2321 and
$(2321)^{\pi}$ = 0102 where π = (3120) $\in$ N
[012304]=[0102]

012303 = (23)(014)32421 and $(32421)^{\pi}$ =
01213 where π = (42130) $\in$ N
[012303]=[01213]

012301 = (02134)230324 and $(230324)^{\pi}$ =
012103 where π = (02)(143) $\in$ N
[012301]=[012103]

[01213] $N^{(0\ 1\ 2\ 1\ 3)}$ = <*e*> has orbits {0}, {1}, {2}, 120
{3}, {4}, 012132 = 01(132)(04)231 =
(132)(04)43231 and $(43231)^{\pi}$ = 01213 where
π = (13)(04) $\in$ N
[012132]=[01213]

	012130 = 01(20143)3124 = (204143)143124 = (1304)13424 = (10)(234)43104 and $(43104)^{\pi}$ = 01230 where π = (12403) $\in$ N	
[012130]=[01230]		
[012131]	$N^{(0\ 1\ 2\ 1\ 3\ 1)}$ = <*e*> has orbits {0}, {1}, {2}, {3}, {4}, 0121313 = (032)403042 and $(403042)^{\pi}$ = 012103 where π = (140)(23) $\in$ N	120
[0121313]=[012103]		
	0121312 = (04)(23)432313 and $(432313)^{\pi}$ = 012131 where π = (13)(40) $\in$ N	
[0121312]=[01213]		
	0121314 = (014)03020 and $(03020)^{\pi}$ = 01020 where π = (13) $\in$ N	
[0121314]=[01020]		
[0121310]	$N^{(0\ 1\ 2\ 1\ 3\ 1\ 0)}$ = <(12034)> has orbit {0, 1, 2, 3, 4}, 0121310 ~ 3202423 ~ 4030104 ~ 1343231 ~ 2414042 and for all k in {0, 1, 2, 3, 4}	24
[0121310k]=[01213]		

The Cayley graph of $P\Sigma L_2(49)$ over S_5 has as vertices the set of right cosets of N in G, i.e. the set Nw_i, where w_i are words in T. The diagram below illustrates each orbit of N in its action on the vertices by right multiplication which is represented by a single node, labeled with the number of vertices that it contains. Lines in between these nodes are labeled with integers to denote how many edges from a vertex of one node will lead to the vertices of the other. At this point we can calculate the action of the five symmetric generators on the cosets of $P\Sigma L_2(49)$ over S_5.

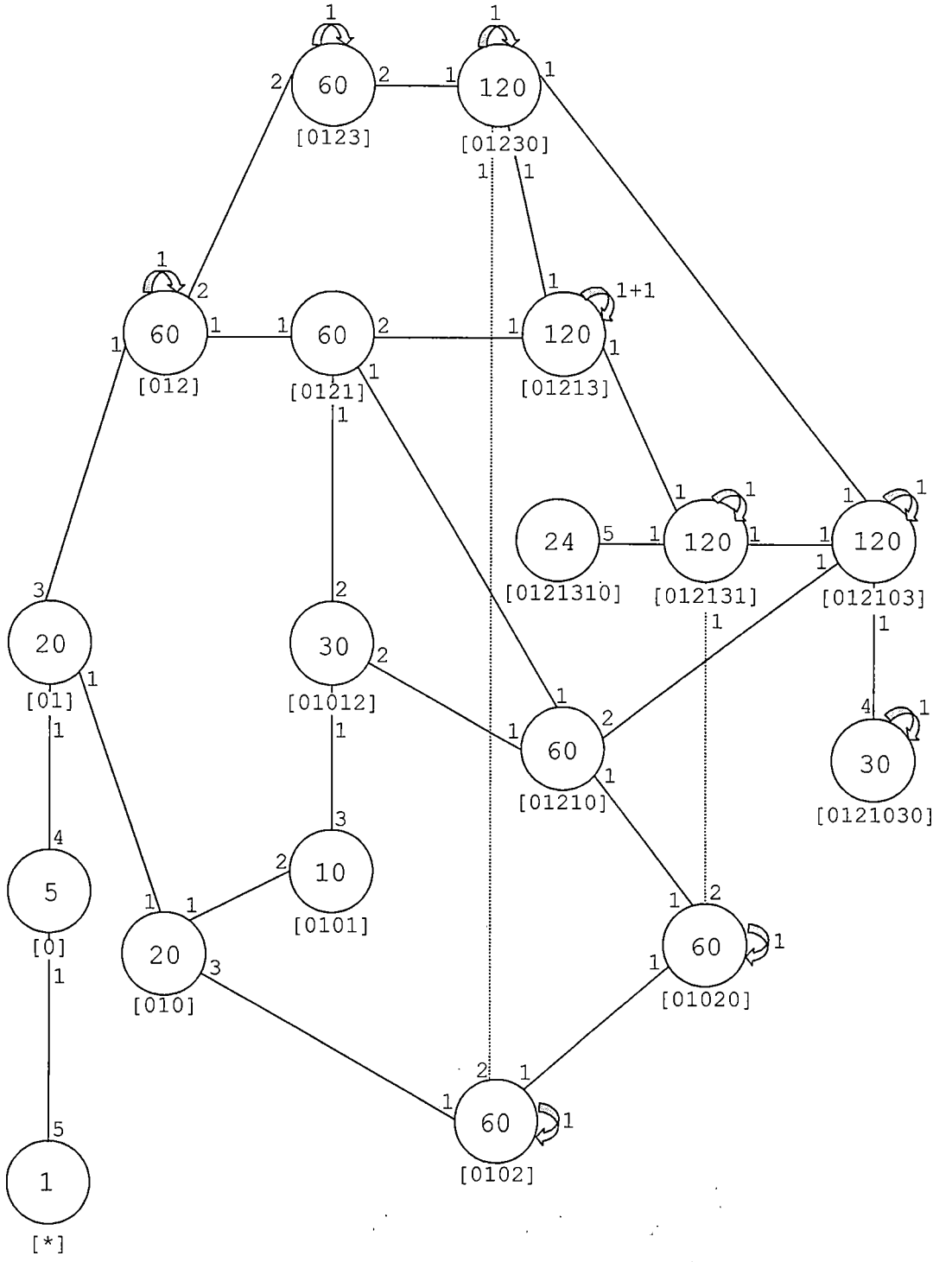

Figure 9.1 Cayley Graph of $P\Sigma L_2(49)$ over S_5

The action of the symmetric generators on the cosets of N in G is given by (see Appendix D(3) for the labeling):

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(368, 525)(370, 528)(373, 429)(375, 534)(376, 535)

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92)(85, 171)(88, 368)(90, 246)(91, 102)(93, 701)(95,
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(237, 481)(238, 549)(240, 752)(241, 495)(242, 524)
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(261, 473)(262, 545)(263, 665)(264, 550)(265, 437)
(267, 558)(268, 711)(270, 622)(271, 611)(273, 444)
(275, 683)(277, 460)(279, 717)(281, 774)(282, 540)
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(220, 673)(222, 339)(224, 682)(226, 362)(229, 471)
(231, 691)(232, 654)(235, 512)(237, 696)(242, 527)
(243, 589)(244, 486)(245, 704)(246, 367)(247, 460)
(248, 676)(250, 537)(251, 692)(254, 626)(255, 571)
(256, 593)(258, 563)(260, 724)(261, 286)(262, 573)
(263, 519)(265, 620)(267, 444)(269, 395)(270, 294)
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(501, 861)(505, 623)(506, 823)(507, 513)(509, 754)
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(520, 551)(521, 528)(522, 739)(525, 862)(526, 747)
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(553, 820)(556, 856)(557, 805)(559, 730)(565, 949)
(566, 627)(567, 843)(568, 655)(570, 740)(572, 871)
(577, 768)(581, 830)(583, 955)(584, 698)(585, 818)
(586, 723)(588, 957)(591, 798)(592, 651)(595, 755)
(601, 756)(602, 879)(603, 838)(604, 842)(605, 844)
(606, 953)(607, 703)(608, 882)(609, 668)(610, 827)
(612, 658)(613, 918)(614, 741)(615, 664)(617, 778)
(625, 811)(630, 834)(631, 969)(632, 715)(635, 689)
(637, 970)(638, 670)(639, 693)(640, 706)(642, 934)
(644, 732)(645, 869)(646, 720)(647, 773)(650, 875)
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(779, 821)(780, 898)(782, 939)(783, 874)(785, 855)
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(810, 972)(813, 962)(816, 819)(817, 929)(825, 960)
(828, 942)(833, 881)(837, 889)(840, 888)(849, 921)
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(873, 956)(878, 974)(883, 910)(887, 938)(890, 907)
(892, 963)(894, 966)(895, 904)(896, 948)(897, 971)
(900, 965)(901, 964)(903, 928)(908, 916)(913, 979)
(919, 968)(930, 977)(935, 959)(947, 973)(950, 954)

Proof of Isomorphism

Double coset enumeration as displayed in Figure 9.1 shows that the group defined by the symmetric presentation contains a homomorphic image of $N \cong S_5$ to index at most

$$\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(0\ 1)}|} + \frac{|N|}{|N^{(0\ 1\ 0)}|} + \frac{|N|}{|N^{(0\ 1\ 0\ 2)}|} +$$

$$\frac{|N|}{|N^{(0\ 1\ 0\ 1)}|} + \frac{|N|}{|N^{(0\ 1\ 0\ 1\ 2)}|} + \frac{|N|}{|N^{(0\ 1\ 2\ 1\ 0)}|} + \frac{|N|}{|N^{(0\ 1\ 0\ 2\ 0)}|} +$$

$$\frac{|N|}{|N^{(0\ 1\ 2)}|} + \frac{|N|}{|N^{(0\ 1\ 2\ 1)}|} + \frac{|N|}{|N^{(0\ 1\ 2\ 3)}|} + \frac{|N|}{|N^{(0\ 1\ 2\ 3\ 0)}|} +$$

$$\frac{|N|}{|N^{(0\ 1\ 2\ 1\ 3)}|} + \frac{|N|}{|N^{(0\ 1\ 2\ 1\ 3\ 1)}|} + \frac{|N|}{|N^{(0\ 1\ 2\ 0\ 3)}|} +$$

$$\frac{|N|}{|N^{(0\ 1\ 2\ 1\ 0\ 3\ 0)}|} + \frac{|N|}{|N^{(0\ 1\ 2\ 1\ 3\ 1\ 0)}|}$$

$$= \frac{120}{120} + \frac{120}{24} + \frac{120}{6} + \frac{120}{6} + \frac{120}{2} + \frac{120}{12} + \frac{120}{4} + \frac{120}{2} +$$

$$\frac{120}{2} + \frac{120}{2} + \frac{120}{2} + \frac{120}{2} + \frac{120}{1} + \frac{120}{1} + \frac{120}{1} + \frac{120}{1} +$$

$$\frac{120}{4} + \frac{120}{5}$$

$$= 1 + 5 + 20 + 20 + 50 + 10 + 30 + 60 + 60 + 60 + 60 + 60 +$$

$$120 + 120 + 120 + 120 + 30 + 24$$

$$= 980.$$

As a result, the index of N in G is at most 980.

Therefore, the order of the image group G is at most

$$980 \cdot |N| = 980 \cdot 120 = 11760.$$

Moreover, the action by right multiplication on the 980 cosets is implicit in the enumeration, and we can verify that the image is the projective sigma linear group

$P\Sigma L_2(49)$.

Furthermore, the order of G can be confirmed by regarding G as a permutation group on the 980 cosets that we have found.

The action of x and y on the single cosets of N in G is given by:

x: (2, 3, 5, 8, 4)(6, 10, 17, 21, 11)(7, 12, 18, 25, 13)(9, 15, 28, 31, 16)(14, 26, 22, 39, 27)(19, 34, 58, 62, 35)(20, 36, 63, 67, 37)(23, 41, 73, 76, 42)(24, 43, 77, 82, 44)(29, 49, 90, 94, 50)(30, 51, 59, 97, 52)(32, 54, 100, 103, 55)(33, 56, 104, 108, 57)(38, 68, 95, 122, 69)(40, 71, 124, 127, 72)(45, 83, 78, 133, 84)(46, 85, 105, 144, 86)(47, 87, 91, 138, 80)(48, 88, 147, 149, 89)(53, 98, 64, 115, 99)(60, 81, 139, 176, 111)(61, 112, 177, 181, 113)(65, 117, 187, 131, 75)(66, 118, 189, 194, 119)(70, 123, 145, 159, 107)(74, 129, 208, 212, 130)(79, 135, 218, 222, 136)(92, 151, 141, 226, 152)(93, 153, 242, 246, 154)(96, 157, 250, 253, 158)(101, 163, 238, 150, 164)(102, 165, 137, 223, 166)(106, 170, 267, 269, 171)(109, 173, 272, 225, 140)(110, 121, 197, 259, 174)(114, 182, 285, 289, 183)(116, 185, 291, 205, 186)(120, 195, 190, 296, 196)(125, 201, 310, 314, 202)(126, 203, 209, 317, 204)(128, 206, 318, 321, 207)(132, 214, 315, 303, 215)(134, 216, 329, 333, 217)(142, 227, 155, 247, 228)(143, 229, 219, 335, 230)(146, 232, 330, 340, 233)(148, 235, 239, 199, 236)(156, 248, 370, 373, 249)(160, 255, 342, 274, 198)(161, 256, 286, 381, 257)(162, 258, 311, 301, 192)(167, 263, 243, 283, 264)(168, 265, 392, 266, 169)(172, 270, 178, 278, 271)(175, 275, 402, 404, 276)(179, 280, 295, 188, 252)(180, 281, 410, 415, 282)(184, 290, 382, 421, 288)(191, 298, 433, 436, 299)(193, 302, 439, 360, 240)(200, 308, 423, 448, 309)(210, 262, 389, 465, 323)(211, 324, 466, 470, 325)(213, 326, 339, 221, 327)(220, 337, 486, 489, 338)(224, 344, 496, 442, 304)(231, 351, 397, 480, 352)(234, 355, 254, 378, 346)(237, 357, 349, 368, 358)(241, 361, 514, 516, 362)(244, 365, 293, 428, 354)(245, 366, 520, 524, 367)(251, 375, 533, 536, 376)(260, 385, 305, 443, 386)(261, 387, 548, 549, 388)(268, 394, 556, 558, 395)(273, 400, 563, 449, 401)

(277, 405, 572, 576, 406)(279, 408, 578, 429, 294)(284, 417, 521, 592, 418)(287, 356, 509, 595, 420)(292, 426, 519, 364, 427)(297, 431, 603, 607, 432)(300, 437, 555, 393, 438)(306, 444, 396, 559, 445)(307, 446, 623, 593, 419)(312, 450, 379, 539, 451)(313, 452, 629, 632, 453) (316, 456, 634, 469, 457)(319, 458, 544, 384, 459)(320, 460, 348, 502, 461)(322, 347, 501, 564, 463)(328, 473, 636, 512, 359)(331, 476, 655, 658, 477)(332, 478, 513, 662, 479)(334, 482, 663, 667, 483)(336, 485, 669, 648, 471)(341, 492, 659, 547, 493)(343, 495, 590, 618, 441) (345, 498, 454, 570, 499)(350, 503, 686, 566, 504)(353, 507, 664, 694, 508)(363, 517, 677, 701, 518)(369, 526, 411, 581, 527)(371, 529, 710, 567, 530)(372, 468, 487, 580, 409)(374, 531, 713, 714, 532)(377, 430, 602, 586, 537)(380, 412, 434, 610, 540)(383, 542, 604, 612, 543) (390, 551, 573, 550, 552)(391, 553, 656, 584, 413)(398, 490, 670, 611, 435)(399, 561, 619, 733, 562)(403, 568, 737, 742, 569)(407, 577, 730, 622, 575)(414, 585, 749, 724, 546)(416, 588, 753, 665, 589)(422, 596, 467, 645, 597)(424, 599, 500, 685, 545)(425, 600, 481, 649, 472) (440, 617, 774, 752, 587)(447, 624, 626, 538, 616)(455, 608, 764, 784, 633)(462, 640, 630, 646, 641)(464, 642, 792, 795, 643)(474, 651, 591, 754, 652)(475, 653, 601, 758, 654)(484, 668, 687, 805, 666)(488, 639, 790, 817, 673) (491, 674, 743, 574, 675)(494, 678, 683, 598, 571) (497, 681, 820, 807, 682)(505, 690, 804, 732, 691)(506, 692, 534, 615, 661)(510, 695, 627, 689, 696)(511, 697, 711, 705, 523)(515, 698, 583, 748, 699)(522, 703, 606, 761, 704)(525, 706, 727, 745, 579)(528, 709, 637, 788, 676)(535, 715, 631, 783, 716)(541, 720, 756, 762, 721) (554, 680, 557, 700, 620)(560, 729, 838, 763, 717)(565, 735, 707, 840, 736)(582, 605, 760, 625, 747)(594, 746, 873, 875, 755)(609, 766, 892, 895, 767)(613, 771, 888, 864, 772)(614, 773, 898, 879, 751)(621, 777, 821, 815, 672)(628, 780, 904, 894, 781)(635, 786, 739, 837, 787) (638, 769, 896, 911, 789)(644, 796, 916, 918, 797)(647, 800, 907, 920, 801)(650, 802, 921, 880, 803)(657, 808, 679, 819, 809)(660, 811, 874, 845, 812)(671, 779, 903, 930, 814)(684, 824, 922, 806, 740)(688, 825, 851, 722, 826)(693, 828, 912, 791, 829)(702, 834, 799, 919, 835) (708, 841, 725, 853, 830)(712, 842, 770, 897, 843)(718, 847, 940, 852, 723)(719, 848, 932, 823, 827)(726, 831, 836, 850, 854)(728, 855, 738, 865, 856)(731, 858, 906, 782, 859)(734, 860, 793, 914, 861)(741, 868, 950, 949, 863)(744, 870, 844, 887, 871)(750, 878, 956, 955, 876)

(757, 884, 881, 951, 869)(759, 885, 813, 928, 886)(765, 833, 934, 913, 891)(768, 857, 941, 952, 893)(775, 900, 964, 915, 794)(776, 901, 862, 839, 832)(778, 899, 872, 954, 902)(785, 866, 935, 970, 909)(798, 883, 958, 933, 917)(810, 925, 849, 943, 926)(816, 924, 974, 977, 929)(818, 877, 846, 939, 923)(822, 867, 946, 966, 905)(882, 890, 961, 973, 957)(889, 910, 971, 938, 953)(908, 968, 936, 959, 969)(931, 937, 948, 965, 978)(942, 972, 975, 947, 980)(944, 979, 962, 976, 960)

y: (3, 5)(6, 9)(10, 18)(12, 22)(15, 17)(16, 27)(19, 33)(23, 40)(25, 39)(26, 28)(29, 32)(30, 47)(34, 59)(35, 50)(36, 64)(41, 63)(43, 78)(44, 80)(45, 70)(46, 48)(49, 91)(51, 95)(54, 58)(55, 84)(56, 105)(57, 107)(60, 110)(61, 93)(65, 116)(67, 115)(68, 100)(69, 89)(71, 73)(72, 99)(74, 114)(77, 85)(79, 134)(81, 137)(82, 122)(83, 104)(87, 145)(88, 90)(92, 150)(94, 144)(96, 156)(97, 159)(98, 124)(101, 109)(102, 142)(103, 149)(106, 169)(111, 171)(112, 178)(118, 190)(119, 192)(120, 184)(121, 148)(123, 147)(125, 128)(126, 162)(129, 209)(130, 202)(133, 138)(135, 219)(139, 173)(140, 198)(141, 168)(146, 231)(151, 239)(152, 164)(153, 243)(154, 230)(155, 170)(157, 177)(158, 233)(160, 254)(161, 200)(163, 259)(165, 228)(166, 236)(167, 237)(174, 274)(175, 268)(176, 226)(179, 279)(181, 283)(182, 286)(183, 288)(185, 187)(186, 291)(189, 256)(191, 297)(193, 300)(194, 303)(195, 285)(196, 207)(197, 255)(199, 234)(201, 311)(203, 315)(206, 208)(210, 322)(211, 313)(214, 318)(215, 309)(216, 330)(217, 271)(218, 248)(220, 336)(222, 340)(223, 342)(224, 273)(225, 266)(227, 346)(229, 349)(232, 242)(235, 269)(238, 355)(240, 359)(241, 261)(244, 364)(245, 350)(246, 368)(247, 272)(249, 264)(250, 351)(251, 277)(252, 354)(258, 382)(260, 384)(262, 348)(263, 329)(265, 267)(270, 397)(275, 319)(276, 362)(278, 335)(281, 411)(282, 413)(284, 416)(287, 419)(290, 423)(292, 294)(293, 425)(295, 427)(296, 301)(298, 434)(302, 344)(304, 441)(305, 361)(306, 320)(307, 379)(308, 310)(312, 449)(314, 381)(316, 455)(317, 421)(321, 448)(323, 420)(324, 467)(326, 408)(327, 472)(328, 447)(331, 475)(332, 399)(333, 480)(334, 371)(337, 487)(338, 477)(339, 428)(341, 491)(343, 494)(345, 393)(347, 500)(352, 358)(353, 363)(356, 396)(357, 370)(360, 388)(366, 521)(367, 523)(369, 525)(372, 391)(374, 505)(375, 534)(376, 530)(378, 392)(383, 541)(385, 545)(386, 401)(387, 454)(389, 402)(390, 474)(394, 557)(395, 512)(398, 560)(400, 564)(403, 515)(404, 570)(405, 573)(406,

575)(407, 484)(410, 490)(412, 582)(414, 583)(415, 586)
(417, 572)(418, 532)(424, 598)(426, 481)(430, 601)(431,
604)(432, 597)(433, 608)(436, 612)(437, 558)(438, 445)
(439, 443)(440, 497)(442, 593)(444, 620)(446, 548)(450,
626)(451, 459)(452, 630)(453, 540)(456, 466)(457, 543)
(458, 636)(460, 499)(461, 616)(462, 625)(463, 618)(464,
594)(465, 539)(468, 602)(470, 646)(471, 579)(473, 495)
(476, 656)(478, 659)(479, 661)(482, 664)(483, 666)(485,
670)(486, 653)(488, 672)(492, 663)(493, 676)(496, 559)
(498, 683)(501, 556)(502, 590)(503, 687)(504, 562)(506,
511)(507, 513)(508, 691)(509, 514)(510, 528)(516, 700)
(517, 627)(518, 589)(519, 600)(520, 551)(522, 702)(524,
574)(526, 669)(527, 654)(529, 711)(531, 533)(535, 712)
(536, 694)(537, 717)(538, 554)(542, 629)(544, 571)(546,
723)(547, 566)(549, 624)(550, 567)(552, 696)(553, 727)
(555, 599)(561, 730)(563, 680)(565, 734)(568, 738)(569,
740)(576, 701)(577, 591)(578, 649)(580, 745)(581, 584)
(585, 617)(587, 751)(588, 637)(592, 733)(595, 685)(596,
756)(603, 640)(605, 632)(606, 757)(607, 762)(609, 765)
(610, 645)(611, 658)(613, 770)(614, 728)(615, 622)(619,
690)(621, 776)(623, 678)(628, 638)(631, 719)(633, 641)
(634, 720)(635, 644)(639, 684)(642, 793)(643, 781)(647,
799)(650, 679)(651, 804)(652, 675)(655, 729)(657, 807)
(660, 810)(662, 665)(667, 754)(668, 677)(671, 803)(673,
814)(674, 713)(681, 821)(682, 736)(686, 695)(688, 766)
(689, 732)(692, 743)(693, 796)(697, 753)(698, 830)(699,
832)(703, 836)(704, 826)(705, 805)(706, 838)(707, 780)
(708, 806)(709, 710)(714, 788)(715, 844)(716, 829)(718,
846)(721, 760)(722, 782)(724, 789)(725, 779)(726, 786)
(731, 857)(735, 862)(737, 769)(739, 866)(741, 867)(742,
795)(744, 833)(746, 872)(747, 764)(748, 875)(749, 840)
(750, 775)(752, 880)(755, 852)(758, 763)(759, 791)(761,
888)(767, 893)(768, 798)(771, 892)(772, 882)(773, 847)
(774, 853)(777, 873)(778, 818)(783, 907)(785, 849)(787,
891)(790, 792)(794, 905)(797, 917)(800, 916)(801, 909)
(802, 896)(808, 923)(809, 860)(811, 912)(812, 859)(813,
883)(815, 879)(816, 929)(817, 819)(820, 899)(822, 931)
(823, 864)(824, 898)(825, 837)(827, 869)(828, 913)(831,
933)(834, 906)(835, 871)(839, 865)(841, 902)(842, 886)
(843, 854)(845, 932)(848, 941)(850, 887)(851, 925)(855,
921)(856, 861)(858, 918)(863, 915)(868, 878)(870, 952)
(874, 890)(876, 955)(877, 922)(881, 943)(884, 958)(885,
926)(889, 960)(894, 954)(895, 928)(897, 935)(900, 965)
(901, 930)(903, 904)(908, 967)(910, 962)(911, 939)(914,
940)(919, 973)(920, 951)(924, 948)(927, 972)(934, 961)

(936, 953)(937, 949)(938, 945)(942, 979)(944, 975)(946, 974)(947, 968)(950, 966)(956, 978)(957, 970)(959, 971)(963, 976)(964, 977)(969, 980).

As a result, the action of x and y on $\{t_0, t_1, t_2, t_3, t_4\}$ is given by:

x: $(t_1, t_2, t_3, t_4, t_0)$

y: (t_1, t_2).

We note that the order of the product xy equals 4, and so $N = \langle x, y \rangle \cong S_5$.

The action of the symmetric generator t_0 on the single cosets of N in G is given by:

t_0 = (1, 2)(3, 6)(4, 7)(5, 9)(8, 14)(10, 19)(11, 20)(12, 23)(13, 24)(15, 29)(16, 30)(17, 32)(18, 33)(21, 38)(22, 40)(25, 45)(26, 46)(27, 47)(28, 48)(31, 53)(34, 60)(35, 61)(36, 65)(37, 66)(39, 70)(41, 74)(42, 75)(43, 79)(44, 81)(49, 92)(50, 93)(51, 96)(52, 62)(54, 101)(55, 102)(56, 106)(57, 68)(58, 109)(59, 110)(63, 114)(64, 116)(67, 120)(69, 121)(71, 125)(72, 126)(73, 128)(76, 132)(77, 85)(78, 134)(80, 137)(82, 140)(83, 141)(84, 142)(86, 143)(87, 146)(88, 133)(89, 148)(90, 138)(91, 150)(94, 155)(95, 156)(97, 160)(98, 161)(99, 162)(100, 107)(103, 167)(104, 168)(105, 169)(108, 172)(111, 175)(112, 179)(113, 180)(115, 184)(117, 188)(118, 191)(119, 193)(122, 198)(123, 199)(124, 200)(127, 205)(129, 210)(130, 211)(131, 213)(135, 220)(136, 221)(139, 224)(144, 170)(145, 231)(147, 234)(149, 237)(151, 240)(152, 241)(153, 244)(154, 245)(157, 251)(158, 252)(159, 254)(163, 260)(164, 261)(165, 262)(166, 226)(171, 268)(173, 273)(174, 267)(176, 236)(177, 277)(178, 279)(181, 284)(182, 287)(183, 214)(185, 292)(186, 293)(187, 294)(189, 256)(190, 297)(192, 300)(194, 304)(195, 305)(196, 306)(197, 307)(201, 312)(202, 313)(203, 316)(204, 212)(206, 319)(207, 320)(208, 275)(209, 322)(215, 328)(216, 331)(217, 332)(218, 334)(219, 336)(222, 341)(223, 343)(225, 345)(227, 347)(228, 348)(229, 327)

(230, 350)(232, 353)(233, 354)(235, 356)(238, 247)
(239, 359)(242, 363)(243, 364)(246, 369)(248, 371)
(249, 372)(250, 374)(253, 377)(255, 379)(257, 380)
(258, 383)(259, 384)(263, 390)(264, 391)(265, 274)
(266, 393)(269, 396)(270, 398)(271, 399)(272, 355)
(276, 403)(278, 407)(280, 409)(281, 412)(282, 414)
(283, 416)(285, 361)(286, 419)(288, 318)(289, 422)
(290, 424)(291, 425)(295, 430)(296, 308)(298, 434)
(299, 435)(301, 310)(302, 440)(303, 441)(309, 447)
(311, 449)(314, 454)(315, 455)(317, 394)(321, 462)
(323, 464)(324, 468)(325, 469)(326, 471)(329, 474)
(330, 475)(333, 481)(335, 484)(337, 488)(338, 457)
(339, 490)(340, 491)(342, 494)(344, 497)(346, 500)
(349, 472)(351, 505)(352, 506)(357, 510)(358, 511)
(360, 513)(362, 515)(365, 489)(366, 522)(367, 501)
(368, 525)(370, 528)(373, 429)(375, 534)(376, 535)
(378, 538)(381, 387)(382, 541)(385, 546)(386, 547)
(388, 507)(389, 550)(392, 554)(395, 446)(397, 560)
(400, 565)(401, 566)(402, 567)(404, 571)(405, 574)
(406, 492)(408, 579)(410, 428)(411, 582)(413, 583)
(415, 587)(417, 590)(418, 591)(420, 594)(421, 557)
(423, 598)(426, 480)(427, 601)(431, 605)(432, 606)
(433, 609)(436, 613)(437, 614)(438, 615)(439, 616)
(442, 619)(443, 461)(444, 621)(445, 622)(448, 625)
(450, 627)(451, 628)(452, 553)(453, 631)(456, 635)
(458, 637)(459, 638)(460, 639)(463, 509)(465, 599)
(466, 644)(467, 602)(470, 647)(473, 650)(476, 657)
(477, 543)(478, 660)(479, 536)(482, 665)(483, 516)
(485, 671)(486, 642)(487, 672)(493, 677)(495, 679)
(496, 680)(498, 533)(499, 684)(502, 572)(503, 688)
(504, 689)(508, 693)(512, 548)(514, 618)(517, 626)
(518, 674)(519, 584)(520, 695)(521, 702)(523, 556)
(524, 573)(526, 707)(527, 708)(529, 711)(530, 712)
(531, 683)(532, 577)(537, 718)(539, 555)(540, 719)
(542, 722)(544, 570)(545, 723)(549, 725)(551, 686)
(552, 726)(558, 728)(559, 563)(561, 731)(562, 732)
(564, 734)(568, 739)(569, 741)(575, 663)(576, 744)
(578, 729)(580, 746)(581, 600)(585, 750)(586, 751)
(588, 636)(589, 713)(592, 709)(593, 690)(595, 652)
(596, 633)(597, 757)(603, 759)(604, 632)(607, 763)
(608, 765)(610, 768)(611, 769)(612, 770)(617, 775)
(620, 776)(623, 778)(624, 779)(629, 782)(630, 727)
(634, 785)(640, 791)(641, 756)(643, 794)(645, 798)
(646, 799)(648, 784)(649, 655)(651, 805)(653, 793)
(654, 806)(656, 807)(658, 737)(659, 810)(661, 694)

(662, 664)(666, 700)(667, 813)(668, 676)(669, 780)
(670, 803)(673, 816)(675, 685)(678, 818)(681, 822)
(682, 823)(687, 766)(691, 796)(692, 827)(696, 786)
(697, 772)(698, 831)(699, 819)(701, 833)(703, 837)
(704, 724)(705, 804)(706, 839)(710, 733)(714, 801)
(715, 777)(716, 845)(717, 846)(720, 849)(721, 850)
(730, 857)(735, 863)(736, 864)(738, 866)(740, 867)
(742, 861)(743, 869)(745, 872)(747, 874)(748, 876)
(749, 877)(752, 881)(753, 882)(754, 883)(755, 802)
(758, 762)(760, 887)(761, 889)(764, 890)(767, 894)
(771, 797)(773, 868)(774, 899)(781, 905)(783, 908)
(787, 910)(788, 909)(789, 826)(790, 828)(792, 913)
(795, 856)(800, 898)(808, 812)(809, 924)(811, 927)
(814, 929)(815, 903)(817, 832)(820, 853)(821, 931)
(824, 916)(825, 836)(829, 932)(830, 933)(834, 935)
(835, 936)(838, 865)(840, 922)(841, 937)(842, 901)
(843, 938)(844, 873)(847, 878)(848, 942)(851, 944)
(852, 896)(854, 945)(855, 946)(858, 947)(859, 923)
(860, 948)(862, 915)(870, 926)(871, 953)(875, 955)
(879, 904)(880, 943)(884, 959)(885, 952)(886, 930)
(888, 960)(891, 962)(892, 917)(893, 954)(895, 963)
(897, 906)(900, 939)(902, 949)(907, 967)(911, 965)
(912, 972)(914, 973)(918, 968)(919, 940)(920, 961)
(921, 974)(925, 975)(928, 976)(934, 951)(941, 979)
(950, 966)(956, 977)(957, 980)(958, 971)(964, 978)
(969, 970).

Now, we need to verify that our relations hold true in $P\Sigma L_2(49)$.

Looking at our first relation we can see that

$t_0^{(t_0t_2t_1t_0t_2t_1t_0)} = t_2$

$t_1^{(t_0t_2t_1t_0t_2t_1t_0)} = t_0$

$t_2^{(t_0t_2t_1t_0t_2t_1t_0)} = t_1$

$t_3^{(t_0t_2t_1t_0t_2t_1t_0)} = t_4$

$t_4^{(t_0t_2t_1t_0t_2t_1t_0)} = t_3$.

Thus, $t_0t_2t_1t_0t_2t_1t_0$ acts as the permutation (1 0 2)(3 4) on the symmetric generators, that is

$$t_0t_2t_1t_0t_2t_1t_0 = (1\ 0\ 2)(3\ 4)$$

$$\Rightarrow t_0t_1t_2t_0 = (1\ 2\ 0)(3\ 4)t_0t_2t_1.$$

Therefore, our first relation holds true in $P\Sigma L_2(49)$.

Now, looking at our second relation we can see that

$$t_0^{(t_2t_1t_0t_4t_3t_2t_1t_0)} = t_2$$

$$t_1^{(t_2t_1t_0t_4t_3t_2t_1t_0)} = t_3$$

$$t_2^{(t_2t_1t_0t_4t_3t_2t_1t_0)} = t_4$$

$$t_3^{(t_2t_1t_0t_4t_3t_2t_1t_0)} = t_0$$

$$t_4^{(t_2t_1t_0t_4t_3t_2t_1t_0)} = t_1.$$

Thus, $t_2t_1t_0t_4t_3t_2t_1t_0$ acts as the permutation (0 3 1 4 2) on the symmetric generators, that is

$$t_2t_1t_0t_4t_3t_2t_1t_0 = (0\ 3\ 1\ 4\ 2)$$

$$\Rightarrow t_0t_1t_2t_3 = (0\ 3\ 1\ 4\ 2)t_2t_1t_0t_4.$$

Therefore, our second relation holds true in $P\Sigma L_2(49)$.

Now, looking at our third relation we can see that

$$t_0^{(t_1t_0)^2} = t_0$$

$$t_1^{(t_1t_0)^2} = t_1$$

$$t_2^{(t_1t_0)^2} = t_2$$

$t_3^{(t_1t_0)^2} = t_3$

$t_4^{(t_1t_0)^2} = t_4.$

Thus, $(t_1t_0)^2$ acts as the identity on the symmetric generators, that is

$$(t_1t_0)^2 = Id$$

$$\Rightarrow t_0t_1t_0t_1 = t_1t_0t_1t_0.$$

Therefore, our third relation holds true in $P\Sigma L_2(49)$. The group $P\Sigma L_2(49)$ is generated by x, y and t_0, and the relations hold true in $P\Sigma L_2(49)$. As a result, $P\Sigma L_2(49)$ is an image of G, therefore

$$|G| \geq |P\Sigma L_2(49)| = 117600.$$

But from previous we know that $|G| \leq 117600$, and so

$$|G| \leq 117600 = |P\Sigma L_2(49)| \leq G,$$

which proves the isomorphism. Hence, $G \cong P\Sigma L_2(49)$.

APPENDIX A

$U_3(3) : 2$

```
N7:=Stabilizer(N,7);
N71:=Stabilizer(N7,1);
for g in N do if 7^g eq 1 and 1^g eq 7 then
N71:=sub<N|N71,g>; end if; end for;
N712:=Stabilizer(N71,2);
for g in N do if 7^g eq 1 and 1^g eq 7 and 2^g eq 2 then
N712:=sub<N|N712,g>;
end if; end for;
for g in N do if 7^g eq 2 and 1^g eq 5 and 2^g eq 1 then
N712:=sub<N|N712,g>;
end if; end for;
N713:=Stabilizer(N71,3);
for g in N do if 7^g eq 3 and 1^g eq 1 and 3^g eq 7 then
N713:=sub<N|N713,g>;
end if; end for;
N7123:=Stabilizer(N712,3);
for g in N do if 7^g eq 1 and 1^g eq 5 and 2^g eq 7 and
3^g eq 6 then N7123:=sub<N|N7123,g>;
end if; end for;
N7124:=Stabilizer(N712,4);
for g in N do if 7^g eq 7 and 1^g eq 5 and 2^g eq 2 and
4^g eq 3 then N7124:=sub<N|N7124,g>;
end if; end for;
for g in N do if 7^g eq 1 and 1^g eq 3 and 2^g eq 5 and
4^g eq 2 then N7124:=sub<N|N7124,g>;
end if; end for;
N71246:=Stabilizer(N7124,6);
for g in N do if
7^g eq 2 and 1^g eq 7 and 2^g eq 1 and 4^g eq 3 and
6^g eq 4 then N71246:=sub<N|N71246,g>; end if; end for;
print Order(N71246);
168
```

APPENDIX B

$L_2(41)$

(Note: 5 denotes 0, and 6 denotes ∞)

Label	Single Coset
1	[N]
2	[5]
3	[1]
4	[4]
5	[6]
6	[2]
7	[1, 5]
8	[3]
9	[4, 5]
10	[6, 5]
11	[2, 5]
12	[2, 1]
13	[6, 1]
14	[5, 4]
15	[5, 6]
16	[3, 5]
17	[5, 1]
18	[3, 1]
19	[3, 4]
20	[2, 6]
21	[6, 4]
22	[1, 6]
23	[4, 1]
24	[1, 4]
25	[3, 6]
26	[3, 2]
27	[4, 6]
28	[2, 1, 5]
29	[6, 2]
30	[6, 1, 5]
31	[1, 3]
32	[5, 4, 5]
33	[4, 3]
34	[5, 6, 5]
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APPENDIX C

NON-SYMMETRICALLY GENERATED GROUPS

```
(1)
N5:=Stabilizer(N,5);
N51:=Stabilizer(N5,1);
for g in N do if 5^g eq 1 and 1^g eq 5 then
N51:=sub<N|N51, g>; end if; end for;
N512:=Stabilizer(N51,2);
for g in N do if 5^g eq 2 and 1^g eq 1 and 2^g eq 5
then N512:=sub<N|N512,g>; end if; end for;
N5123:=Stabilizer(N512,3);
for g in N do if 5^g eq 2 and 1^g eq 1 and 2^g eq 5 and
3^g eq 3 then N5123:=sub<N|N5123,g>; end if; end for;
for g in N do if 5^g eq 5 and 1^g eq 3 and 2^g eq 2 and
3^g eq 1 then N5123:=sub<N|N5123,g>; end if; end for;
for g in N do if 5^g eq 2 and 1^g eq 3 and 2^g eq 5 and
3^g eq 1 then N5123:=sub<N|N5123,g>; end if; end for;
N51234:=Stabilizer(N5123,4);
for g in N do if 5^g eq 1 and 1^g eq 2 and 2^g eq 3 and
3^g eq 4 and 4^g eq 5 then N51234:=sub<N|N51234,g>;
end if; end for;
print Order(N51234);
60

(2)
N5:=Stabilizer(N,5);
N51:=Stabilizer(N5,1);
for g in N do if 5^g eq 1 and 1^g eq 5 then
N51:=sub<N|N51,g>; end if; end for;
N512:=Stabilizer(N51,2);
for g in N do if 5^g eq 1 and 1^g eq 5 and 2^g eq 2 then
N512:=sub<N|N512,g>; end if; end for;
for g in N do if 5^g eq 2 and 1^g eq 1 and 2^g eq 5 then
N512:=sub<N|N512,g>; end if; end for;
N5123:=Stabilizer(N512,3);
for g in N do if 5^g eq 2 and 1^g eq 1 and 2^g eq 5 and
3^g eq 3 then N5123:=sub<N|N5123,g>; end if; end for;
for g in N do if 5^g eq 1 and 1^g eq 5 and 2^g eq 2 and
3^g eq 3 then N5123:=sub<N|N5123,g>; end if; end for;
for g in N do if 5^g eq 3 and 1^g eq 2 and 2^g eq 1 and
3^g eq 5 then N5123:=sub<N|N5123,g>; end if; end for;
N51234:=Stabilizer(N5123,4);
for g in N do if 5^g eq 1 and 1^g eq 2 and 2^g eq 5 and
3^g eq 4 and 4^g eq 3
then N51234:=sub<N|N51234,g>; end if; end for;
print Order(N51234);
120
```

APPENDIX D

$P\Sigma L_2(49)$

```
(1)
S:={[5,1,2,1,5,3,5]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..120] do
for g in H do if
f(t)*f(t^x)*f(t^(x^2))*f(t^x)*f(t)*f(t^(x^3))*f(t) eq
g*f(t^(x^(Rep(SSS[i])[1])))
 *f(t^(x^(Rep(SSS[i])[2])))
 *f(t^(x^(Rep(SSS[i])[3])))
 *f(t^(x^(Rep(SSS[i])[4])))
 *f(t^(x^(Rep(SSS[i])[5])))
 *f(t^(x^(Rep(SSS[i])[6])))
 *f(t^(x^(Rep(SSS[i])[7])))
then print SSS[i];
end if; end for; end for;

(2)
S:={[5,1,2,1,3,1,5]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..120] do
for g in H do if
f(t)*f(t^x)*f(t^(x^2))*f(t^x)*f(t^(x^3))*f(t^x)*f(t) eq
g*f(t^(x^(Rep(SSS[i])[1])))
 *f(t^(x^(Rep(SSS[i])[2])))
 *f(t^(x^(Rep(SSS[i])[3])))
 *f(t^(x^(Rep(SSS[i])[4])))
 *f(t^(x^(Rep(SSS[i])[5])))
 *f(t^(x^(Rep(SSS[i])[6])))
 *f(t^(x^(Rep(SSS[i])[7])))
then print SSS[i];
end if; end for; end for;

(3)
Note: 5 denotes 0
```

Label	Single Coset		
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4	[4]	8	[3]

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